Key

Homework Set 9

(sect 6.3: Partial Fractions)

Write out the partial fraction decomposition for the given fractions. Do not integrate. For extra credit, find the values of the coefficients on the smaller fractions.

1. $\frac{2x}{x^3+x^2} = \frac{2x}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} = \frac{2}{x} - \frac{2}{x+1}$

$A = 2$
$B = 0$
$C = -2$

2. $\frac{x^2+2x+1}{x^3+x} = \frac{x^2+2x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{1}{x} + \frac{2}{x^2+1}$

$A = 1$
$B = 0$
$C = 2$

Evaluate the given integrals.

3. $\int \frac{x}{x-6}dx$

$= \int \frac{x-6}{x-6} + \frac{6}{x-6}\,dx = \int 1 + \frac{6}{x-6}\,dx$

$= x + 6\ln(x-6) + C$

4. $\int \frac{dx}{(x+4)(x-1)} = \int \frac{A}{x+4} + \frac{B}{x-1}\,dx = A\ln(x+4) + B\ln(x-1) = -\frac{1}{5}\ln(x+4) + \frac{1}{5}\ln(x-1) + C$

$1 = A(x-1) + B(x+4)$

$x = 1 \Rightarrow \quad 1 = 5B, \quad B = 1/5$

$x = -4 \Rightarrow \quad 1 = -5A, \quad A = -1/5$

OR $= \ln \sqrt[5]{\frac{x-1}{x+4}} + C$

5. $\int_0^1 \frac{2x+3}{(x+1)^2}dx = \int_0^1 \frac{A}{x+1} + \frac{B}{(x+1)^2}\,dx = \int_0^1 \frac{2}{x+1} + \frac{1}{(x+1)^2}\,dx$

$2x+3 = A(x+1) + B$

$x = -1, \quad 1 = B$

$x = 0, \quad 3 = A + B$

$2 = A$

$= 2\ln(x+1) - \frac{1}{x+1} \Big|_0^1$

$= (2\ln 2 - 1/2) - (2\ln 1 - 1) = 2\ln 2 + 1/2$

6. $\int \frac{x^2+2x-1}{x^3-x}dx$

$= \int \frac{x^2+2x-1}{x(x-1)(x+1)}dx = \int \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}\,dx = A\ln x + B\ln(x-1) + C\ln(x+1)$

$= \ln x + \ln(x-1) - \ln(x+1) + C$

$x^2+2x-1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$

$x = 0, \quad -1 = -A, \quad A = 1$

$x = 1, \quad 2 = 2B, \quad B = 1$

$x = -1, \quad -2 = 2C, \quad C = -1$

OR $= \ln\left(\frac{x(x-1)}{x+1}\right) + C$

7. $\int \frac{5}{(x+1)(x^2+4)}dx = \int \frac{A}{x+1} + \frac{Bx+C}{x^2+4}\,dx = \int \frac{1}{x+1} - \frac{x}{x^2+4} + \frac{1}{x^2+4}\,dx$

$5 = A(x^2+4) + Bx(x+1) + C(x+1)$

$x=-1,\quad 5 = 5A,\quad A=1$

$x=0,\quad 5 = 4 + C,\quad C=1$

$x=1,\quad 5 = 5 + 2B + 2,\quad B=-1$

$= \int \frac{1}{1+x} - \frac{1}{2}\cdot\frac{2x}{x^2+4} + \frac{1}{x^2+4}\,dx$

$= \ln(x+1) - \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$

8. $\int \frac{x}{x^4-1}dx = \int \frac{x}{(x^2+1)(x-1)(x+1)}dx = \int \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}\,dx = \int \frac{1}{4}\cdot\frac{1}{x+1} + \frac{1}{4}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{x}{x^2+1}\,dx$

$x = A(x-1)(x^2+1) + B(x+1)(x^2+1) + Cx(x+1)(x-1) + D(x^2-1)$

$x=1\quad 1 = 4B,\quad B = 1/4$

$x=-1\quad -1 = -4A,\quad A = 1/4$

$x=0\quad 0 = -\frac{1}{4} + \frac{1}{4} + -D,\quad D=0$

$x=2\quad 2 = 5/4 + 15/4 + 6C \Rightarrow -3 = 6C$

$-\frac{1}{2} = C$

$\downarrow$

$= \frac{1}{4}\ln(x+1) + \frac{1}{4}\ln(x-1) - \frac{1}{4}\ln(x^2+1) + C$

OR $= \ln \sqrt[4]{\frac{x^2-1}{x^2+1}} + C$

9. $\int \frac{2x-1}{x^2-4x+13}dx$

$= \int \frac{2x-4}{x^2-4x+13} + \frac{3}{x^2-4x+13}\,dx$

$= \ln(x^2-4x+13) + \int \frac{3}{(x-4x+4)+9}\,dx$

$= \ln(x^2-4x+13) + \int \frac{3}{(x-2)^2+3^2}\,dx$

$= \ln(x^2-4x+13) + \arctan\left(\frac{x-2}{3}\right) + C$

10. $\int \ln(x^2-2x+5)\,dx$

$f(x) = \ln(x^2-2x+5)\qquad f'(x) = \frac{2x-2}{x^2-2x+5}$

$g'(x) = 1\qquad g(x) = x$

$= x\ln(x^2-2x+5) - \int \frac{2x^2-2x}{x^2-2x+5}\,dx$

$= x\ln(x^2-2x+5) - \int 2 + \frac{2x-10}{x^2-2x+5}\,dx$

$$\begin{array}{r|l} & \;2 \\ x^2-2x+5 & \;2x^2-2x+0 \\ & \;\underline{2x^2-4x+10} \\ & \;\;2x-10 \end{array}$$

$= x\ln(x^2-2x+5) - 2x - \int \frac{2x-2}{x^2-2x+5} - \frac{8}{x^2-2x+5}\,dx$

$2x-10 = (2x-2) - 8$

$= x\ln(x^2-2x+5) - 2x - \ln(x^2-2x+5) + \int \frac{8}{x^2-2x+5}\,dx$

$= (x-1)\ln(x^2-2x+5) - 2x + \int \frac{8}{(x^2-2x+1)+4}\,dx$

$= (x-1)\ln(x^2-2x+5) - 2x + 4\int \frac{2}{(x-1)^2+2^2}\,dx$

$= (x-1)\ln(x^2-2x+5) - 2x + 4\arctan\left(\frac{x-1}{2}\right) + C$