WeBWorK assignment number Sec8.6 is due : 11/30/2010 at 11:30pm EST.

Differentiation and Integration of Power Series:

Let $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$. ~~Then~~ $C_0 + C_1(x-a) + C_2(x-a)^2 + \dots$

A. $f'(x)=\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ $= C_1 + 2C_2(x-a)^1 + \dots$

B. $\int f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C.$

C. These 3 power series have the same radius of convergence.

Finding new series from old:

$$\boxed{\frac{1}{1-x} = 1+x+x^2+x^3+\dots = \sum_{n=0}^{\infty} x^n, \quad |x|<1, R=1.}$$ $I=(-1,1)$

Replace x by $-x^2$ and get: $\frac{1}{1+x^2} = 1-x^2+x^4-x^6+\dots = \sum_{n=0}^{\infty}(-1)^n x^{2n}, \quad R=1.$

Integrate and get: $\tan^{-1}x = \int \frac{1}{1+x^2} = \boxed{x-\frac{x^3}{3}+\frac{x^5}{5}-\dots} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}, \quad R=1.$

Also $\ln(1-x) = \int \frac{-dx}{1-x} = -\int(1+x+x^2+x^3+\dots)dx = -x-\frac{x^2}{2}-\frac{x^3}{3}-\dots = -\sum_{n=1}^{\infty}\frac{x^n}{n}, \quad R=1.$

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1. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.6.3.pg

Book Problem 3

$\frac{1}{1-x} = 1+x+x^2+\dots, \quad |x|<1$

Find a power series representation for the following function:

replace x by $-3x$

$f(x) = \frac{1}{1+3x} = c_0 + c_1x + c_2x^2 + \dots$, where

$= \frac{1}{1-(-3x)} = 1+(-3x)+(-3x)^2+\dots = 1-3x+9x^2-\dots$

$c_0 =$ 1,
$c_1 =$ -3,
$c_2 =$ 9.

$C_0 =$ constant
$C_1 =$ coefficient of x
$C_2 =$ " " " " x^2

The radius of convergence $R =$ 1/3

$|-3x|<1$

$|3x|<1$

$|x|<\frac{1}{3}$

Another way of writing this power series is : [?]

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$\frac{1}{1+3x} = \sum_{n=0}^{\infty}(-3x)^n$

A) $\frac{1}{1+3x} = \sum_{n=0}^{\infty} 3x^n$

B) $\frac{1}{1+3x} = \sum_{n=0}^{\infty} 3^n x^n.$

[✓ C)] $\frac{1}{1+3x} = \sum_{n=0}^{\infty}(-1)^n 3^n x^n.$

2. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.6.5.pg

Book Problem 5

$\frac{1}{1-x} = 1 + x + x^2 + x^3$

Find a power series representation for the following function:

$\frac{1}{1-x^2} = 1 + x^2 + (x^2)^2 + (x^2)^3 + \cdots$

$f(x) = \frac{1}{1-x^2} = c_0 + c_1 x + c_2 x^2 + \ldots$, where

$= 1 + x^2 + x^4 + x^6 + - - -$

$c_0 = \underline{1}$,
$c_1 = \underline{0}$,
$c_2 = \underline{1}$,
$c_3 = \underline{0}$,
$c_4 = \underline{1}$, $c_5 = \underline{0}$.

same as $= 1 + 0x + 1x^2 + 0x^3 + 1x^4 + \ldots$

$|x^2| < 1 \Rightarrow |x| < 1$

The radius of convergence $R = \underline{1}$.

Another way of writing this power series is : ?

or $\frac{1}{1-x} = \Sigma x^n$

$\frac{1}{1-x^2} = \Sigma (x^2)^n = \Sigma x^{2n}$

[✓ A)] $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$ $= 1 + x^2 + x^4 + - - -$

B) $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

C) $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2^n}$.

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3. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.6.7.pg

Book Problem 7

$\frac{1}{1-x} = 1 + x + x^2 \cdots$, $\frac{1}{x-6} = \frac{-1}{6-x} = \frac{-1}{6} \cdot \frac{1}{1-\frac{x}{6}}$

Find a power series representation for the following function:

$= -\frac{1}{6}\left(1 + \frac{x}{6} + \frac{x^2}{36} \cdots\right)$

$\frac{1}{x-6} = c_0 + c_1 x + c_2 x^2 + \ldots$, where

$= -\frac{1}{6} - \frac{x}{36} - \frac{x^2}{216} \cdots$

$c_0 = \underline{-1/6}$,
$c_1 = \underline{-1/36}$,
$c_2 = \underline{\quad}$,
$c_3 = \underline{\quad}$,

$\left|\frac{x}{6}\right| < 1$

$|x| < 6$

The radius of convergence $R = \underline{6}$.

Another way of writing this power series is : ?

A) $\frac{1}{x-6} = \sum_{n=0}^{\infty} \frac{x^n}{6^n}$

B) $\frac{1}{x-6} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{6^n}$.

C) $\frac{1}{x-6} = \sum_{n=0}^{\infty} -\frac{x^n}{6^{n+1}}$. ✓ $= -\frac{1}{6} - \frac{x}{6^2} \cdots$

$|x|<1, R=1$

4. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.6.9.pg

Book Problem 9 $\frac{1}{1-x} = 1+x+x^2+\cdots$

Find a power series representation for the following function:

$f(x) = \frac{x}{8+x^2} = \frac{x}{8} - \frac{x^3}{64} + \frac{x^5}{512} + \cdots$ (enter only the first 3 non-zero terms).

The radius of convergence $R = 2\sqrt{2}$. $\left|-\frac{x^2}{8}\right|<1 \Rightarrow$

$$\frac{x}{8+x^2} = \frac{x}{8}\cdot\frac{1}{1+\frac{x^2}{8}} = \frac{x}{8}\cdot\boxed{\frac{1}{1-\left(-\frac{x^2}{8}\right)}}$$

replace x by $-\frac{x^2}{8}$

$$= \frac{x}{8}\left(1+\frac{-x^2}{8}+\left(\frac{-x^2}{8}\right)^2+\cdots\right)$$

$$= \frac{x}{8}-\frac{x^3}{64}+\frac{x^5}{8^3}-\cdots$$

$$|x^2|<8 \Rightarrow |x|<\sqrt{8} = 2\sqrt{2} = R$$

5. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.6.11.pg

Book Problem 11

Follow the steps below to find a power series representation for $f(x) = \frac{20}{x^2+3x-4}$:

$\frac{20}{x^2+3x-4} = \frac{A}{x-1}+\frac{B}{x+4}$, where A = 4 and B = -4

$$\frac{20}{x^2+3x-4} = \frac{A}{x-1}+\frac{B}{x+4} = \frac{A(x+4)+B(x-1)}{(x-1)(x+4)}$$

$$20 = A(x+4)+B(x-1)$$

Let $x=1$: $20 = 5A \Rightarrow A=4$

Let $x=-4$: $20 = -5B \Rightarrow B=-4$

Find the first 4 non-zero terms in the power series representation of the following fractions:

$\frac{1}{x-1} = -1-x-x^2-x^3\ldots$

$\frac{1}{x+4} = \frac{1}{4}-\frac{x}{16}+\frac{x^2}{64}-\frac{x^3}{256}+\cdots$

Memory $\boxed{\frac{1}{1-x} = 1+x+x^2+x^3+\cdots}$

$$\frac{1}{x-1} = \frac{-1}{1-x} = -1-x-x^2-x^3\cdots$$

$$\frac{1}{x+4} = \frac{1}{4+x} = \frac{1}{4}\cdot\frac{1}{1+\frac{x}{4}}$$

$$= \frac{1}{4}\cdot\frac{1}{1-\frac{-x}{4}} = \frac{1}{4}\left(1+\frac{-x}{4}+\left(\frac{-x}{4}\right)^2+\left(\frac{-x}{4}\right)^3+\cdots\right)$$

remove x
put $-\frac{x}{4}$

$$\frac{1}{x+4} = \frac{1}{4}-\frac{x}{16}+\frac{x^2}{64}-\frac{x^3}{256}+\cdots$$

Therefore $f(x) = \frac{20}{x^2+3x-4} = c_0+c_1x+c_2x^2+\ldots$, where

$c_0 = -5$

$c_1 = -15/4$

$c_2 = -65/16$

$c_3 =$ ____.

$$\frac{20}{x^2+3x-4} = \frac{4}{x-1} - \frac{4}{x+4}$$

$$= 4\left(\frac{1}{x-1}\right) - 4\left(\frac{1}{x+4}\right)$$

$$= 4(-1-x-x^2-x^3\ldots) - 4\left(\frac{1}{4}-\frac{x}{16}+\frac{x^2}{64}-\frac{x^3}{256}+\cdots\right)$$

$$= -4-4x-4x^2-4x^3\cdots -1+\frac{x}{4}-\frac{x^2}{16}+\frac{x^3}{64}\cdots$$

$$= -5-4x+\frac{x}{4}-4x^2-\frac{x^2}{16}-4x^3+\frac{x^3}{64}\cdots$$

$$= -5-\frac{15}{4}x-\frac{65}{16}x^2-\frac{255}{64}x^3+\cdots$$

6. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.6.13.pg

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots$$

$$\boxed{\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots}$$

Book Problem 13

Follow the steps below to find a power series representation for the function $f(x) = \frac{9x^3}{(1+x)^2}$:

a) A power series for $\frac{1}{1+x}$ = ________ +... (first 4 non-zero terms)

b) Observe that $\frac{1}{(1+x)^2} = \frac{d}{dx}\left(\frac{-1}{1+x}\right)$. $= -\frac{d}{dx}\left(\frac{1}{1+x}\right) = -\left(\frac{1}{1+x}\right)' = -\left(-1 + 2x - 3x^2 - \cdots\right)$

$$= \boxed{1 - 2x + 3x^2 - \cdots}$$

c) A power series for $\frac{1}{(1+x)^2}$ = ________ +... (first 3 non-zero terms)

d) The function $f(x) = \frac{9x^3}{(1+x)^2} = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots$

$$\frac{9x^3}{(1+x)^2} = 9x^3\left(\frac{1}{(1+x)^2}\right) = 9x^3\left(1 - 2x + 3x^2 - \cdots\right)$$

$$= 9x^3 - 18x^4 + 27x^5 - \cdots$$

where

$c_0 = 0$, $c_1 = 0$, $c_2 = 0$, $c_3 = 9$, $c_4 = -18$, $c_5 = 27$.

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7. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.6.15.pg

Book Problem 15

$$\frac{1}{9-x} = \frac{1}{9} \cdot \frac{1}{1-\frac{x}{9}} = \frac{1}{9}\left(1 + \frac{x}{9} + \left(\frac{x}{9}\right)^2 + \cdots\right)$$

$$= \frac{1}{9} + \frac{x}{81} + \frac{x^2}{729} + \cdots$$

Follow the steps below to find a power series representation for the function $f(x) = \ln(9-x)$:

a) A power series for $\frac{1}{9-x} = \frac{1}{9} + \frac{x}{81} + \frac{x^2}{729}$ +... (first 3 non-zero terms)

b) Observe that $\ln(9-x) = \int \frac{-1}{9-x}\,dx = -\int \frac{1}{9-x}dx = -\int \left(\frac{1}{9} + \frac{x}{81} + \frac{x^2}{729} + \cdots\right)dx$

c) The function $f(x) = \ln(9-x)$ = ________ +... (first 3 non-zero terms).

$$\ln(9-x) = -\frac{1}{9}x - \frac{x^2}{162} - \frac{x^3}{2187} + \cdots + C$$

$$\text{If } x = 0 \Rightarrow \ln 9 = 0 \cdots + C \Rightarrow C = \ln 9$$

$$\boxed{f(x) = \ln(9-x) = \ln 9 - \frac{1}{9}x - \frac{x^2}{162} - \frac{x^3}{2187} - \cdots}$$

8. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.6.25.pg

Book Problem 25

Evaluate the indefinite integral $\int \frac{6x-\tan^{-1}(6x)}{x^3}\,dx$ as a power series.

Enter the first 3 non-zero terms in the power series representation of the following functions:

$$\frac{6x-\tan^{-1}(6x)}{x^3} = \frac{6^3}{3} - \frac{6^5x^2}{5} + \frac{6^7x^4}{7} - \cdots +\ldots$$

$$\int \frac{6x-\tan^{-1}(6x)}{x^3} = C + \frac{6^3x}{3} - \frac{6^5x^3}{15} + \frac{6^7x^5}{35} - \cdots$$

The Radius of convergence $R =$ 1/6

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \cdots$$

$$\tan^{-1}(6x) = 6x - \frac{(6x)^3}{3} + \frac{(6x)^5}{5} - \frac{(6x)^7}{7}$$

$$\frac{6x - \tan^{-1}6x}{x^3} = \frac{\cancel{6x} - (\cancel{6x} - \frac{6^3x^3}{3} + \frac{6^5x^5}{5} - \frac{6^7x^7}{7}\cdots)}{x^3}$$

$$= \frac{6^3}{3} - \frac{6^5x^2}{5} + \frac{6^7x^4}{7} - \cdots$$

$$|6x| < 1 \Rightarrow |x| < \frac{1}{6}$$

$$R = \frac{1}{6}$$

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9. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.6.27.pg

Book Problem 27

Use a power series to approximate the definite integral $\int_0^{0.2} \frac{1}{1+x^6}\,dx$ to 6 decimal places.

Enter the first 4 non-zero terms of the power series representation of the following functions:

$$\frac{1}{1+x^6} = 1 + x^6 + x^{12} + x^{18} + \ldots$$

$$\int \frac{1}{1+x^6}\,dx = C + x + \frac{x^7}{7} + \frac{x^{13}}{13} + \frac{x^{19}}{19} + \cdots \Rightarrow$$

Therefore $\int_0^{0.2} \frac{1}{1+x^6}\,dx \approx$ ______ correct to 6 decimal places.

error < 0.000001

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \Rightarrow \frac{1}{1+x^6} = \frac{1}{1-(-x^6)} = 1 + (-x^6) + (-x^6)^2 + (-x^6)^3 + \cdots$$

$$= 1 + x^6 + x^{12} + x^{18} + \cdots$$

$$\int \frac{1}{1+x^6}\,dx = \int (1 + x^6 + \cdots)\,dx$$

$$= x + \frac{x^7}{7} + \frac{x^{13}}{13} - \frac{x^{19}}{19} + \cdots$$

$$\int_0^{0.2} \frac{1}{1+x^6} = 0.2 - \frac{0.2^7}{7} + \frac{0.2^{13}}{13} - \frac{0.2^{19}}{19}$$

$$= \boxed{0.2 - 0.00001828} + \underbrace{0.000000063}_{< 0.00001}$$

$$\| \; 0.199998$$

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10. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.6.29.pg

Book Problem 29

Use a power series to approximate the definite integral $\int_0^{0.1} x\arctan(9x)\,dx$.

Enter the first 3 non-zero terms in the power series representation of the following functions:

$$x\arctan(9x) = 9x^2 - \frac{9^3x^4}{3} + \ldots + \frac{9^5x^6}{5} - \cdots$$ integrate to get

$$\int x\arctan(9x)\,dx = C + 3x^3 - \frac{9^3x^5}{15} + \ldots + \frac{9^5x^7}{35} - \cdots$$

Approximate the definite integral using the first 2 terms only:

$\int_0^{0.1} x\arctan(9x)\,dx \approx$ 0.002514 .(enter 6 digits after the decimal point)

Using the Alternating series test, the error in using the first two terms only to approximate this integral is

$E \leq$ ______.(enter 6 digits after the decimal point)

0.000187

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} \cdots$$

$$x\arctan(9x) = x\left(9x - \frac{9^3x^3}{3} + \frac{9^5x^5}{5} \cdots\right)$$

$$= 9x^2 - \frac{9^3x^4}{3} + \cdots$$

$$\Rightarrow \int_0^{0.1} \arctan(9x)\,dx = \underbrace{3(0.1)^3 - \frac{9^3(0.1)^5}{15}}_{} + \frac{9^5(0.1)^7}{35} \cdots$$

$$= 0.002514$$

error = 0.000187 < 3rd term

$\frac{1}{1-x} = \Sigma x^n$ Ⓓ $\Rightarrow \frac{1}{1-x^2} = \Sigma (x^2)^n = \Sigma x^{2n}$ Ⓑ

$\arctan x = \Sigma (-1)^n \frac{x^{2n+1}}{2n+1}$ Ⓒ $\Rightarrow \arctan 2x = \Sigma (-1)^n \frac{(2x)^{2n+1}}{2n+1}$ Ⓕ

11. (2 pts) UNCC1242/EssentialCalculus-Stewart-Sec8.6.Extra1.pg

MIX and MATCH

B 1. $f(x) = \frac{1}{1-x^2}$

F 2. $f(x) = \arctan 2x$

G 3. $f(x) = \frac{1}{1-2x}$

D 4. $f(x) = \frac{1}{1-x}$

Ⓐ 5. $f(x) = \ln(1-x)$

C 6. $f(x) = \arctan x$

Ⓔ 7. $f(x) = \frac{1}{1+2x}$

Ⓗ 8. $f(x) = \ln(1-2x)$

A. $-\sum_{n=1}^{\infty} \frac{x^n}{n}$

B. $\sum_{n=0}^{\infty} x^{2n}$

C. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

D. $\sum_{n=1}^{\infty} x^n$

E. $\sum_{n=0}^{\infty} (-1)^n 2^n x^n$

F. $\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+1}}{2n+1}$

G. $\sum_{n=0}^{\infty} 2^n x^n$

H. $-\sum_{n=1}^{\infty} \frac{2^n x^n}{n}$

$\frac{1}{1-x} = \Sigma x^n \Rightarrow \frac{1}{1-2x} = \Sigma (2x)^n$ Ⓖ

$\Rightarrow \frac{1}{1+2x} = \Sigma (-2x)^n$ Ⓔ

$$\ln(1-x) = \int \frac{-1}{1-x}\,dx = -\int \frac{1}{1-x}\,dx = -\int \sum_{n=0}^{\infty} x^n\,dx$$

$$= -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

Ⓐ

~~ln(1-2x)~~

$$\int \frac{1}{1-2x}\,dx = \frac{\ln|1-2x|}{-2} + C$$

$$\Rightarrow \ln|1-2x| = -2\int \frac{1}{1-2x}\,dx$$

$$= -2\int \sum_{n=0}^{\infty} (2x)^n\,dx$$

$$= -2 \sum_{n=0}^{\infty} 2^n \frac{x^{n+1}}{n+1}$$

$$= -\sum_{n=0}^{\infty} 2^{n+1} \frac{x^{n+1}}{n+1}$$

$$= -\Sigma\, 2^n \frac{x^n}{n}$$

Ⓗ