Wafaa Shaban **Fall2010-Math1242Common 005**

WeBWorK assignment number Sec8.7 is due : 12/03/2010 at 11:30pm EST.

The **Taylor Series** of f at a is $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...$

The **Maclaurin series** of f is $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$

Some importantMaclaurin series:

$$\frac{1}{1-x} = 1+x+x^2+x^3+.... = \sum_{n=0}^{\infty} x^n, \quad (-1,1).$$

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+.... = \sum_{n=0}^{\infty}\frac{x^n}{n!}, \quad (-\infty,\infty).$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-.... = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}, \quad (-\infty,\infty).$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-.... = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}, \quad (-\infty,\infty).$$

$$\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-.... = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)}, \quad [-1,1].$$

need to memorize all of these formulas

The **Binomial Series** : If k is any real number and $|x|<1$, then $(1+x)^k = 1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+....$

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1. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.7.1.pg

Book Problem 1

$f(x) = f(6) + \frac{f'(6)}{1!}(x-6) + \frac{f''(6)}{2!}(x-6)^2 ..... + \frac{f^{(5)}(6)}{5!}(x-6)^5 + ...$

If $f(x) = \sum_{n=0}^{\infty} b_n(x-6)^n$ for all x, which of the following formulas for b_5 is true? $a=6$

$$\boxed{b_5 = \frac{f^{(5)}(6)}{5!}}$$

Enter T for true and F for false:

___1. $b_5 = \frac{f^{(5)}(0)}{5!}$

T 2. $b_5 = \frac{f^{(5)}(6)}{5!}$

___3. $b_5 = \frac{f^{(5)}(6)}{5}$

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2. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.7.3.pg

$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + ...$

Book Problem 3

If $f^{(n)}(0) = (n+3)!$ for $n = 0,1,2,...,$ then the Maclaurin series for f is

$f(x) =$ ____________ +... (Enter only the first four non-zero terms.)

$f^{(3)}(0) = (3+3)! = 6!$

$f''(0) = (2+3)! = 5!$

$f'(0) = (1+3)! = 4!$

$f(0) = (0+3)! = 3!$

$$f(x) = 3! + \frac{4!}{1!}x + \frac{\overset{5\cdot4\cdot3}{5!}}{2!}x^2 + \frac{\overset{6\cdot5\cdot4}{6!}}{3!}x^3 + ... = 6 + 24x + 60x^2 + 120x^3 + ...$$

3. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.7.4.pg

Book Problem 4 Taylor Series $= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

If $f^{(n)}(6) = \frac{(-1)^n n!}{4^n(n+2)}$ for $n = 0,1,2,\ldots$, then the Taylor series for f centered at 6 is

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(6)}{n!}(x-6)^n$

$= \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{n!}}{4^n(n+2)\cancel{n!}}(x-6)^n$

If $n=0$: $\frac{1}{2}$

If $n=1$: $\frac{-1}{4*3}$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+2)}(x-6)^n.$$

$f(x) = \underline{\frac{1}{2}} + \underline{\frac{-1}{12}}(x-6) + \underline{\frac{1}{64}}(x-6)^2 + \_\_\_ (x-6)^3 + \ldots$

4. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.7.5.pg

Book Problem 5 $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$, $\cos(3x) = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \cdots$

$= 1 - \frac{9x^2}{2} + \frac{81x^4}{24} \cdots$

Find the Maclaurin series for $f(x) = \cos(3x)$ using the definition of a Maclaurin series:

$f(x) = \_\_\_$, $f(0) = \_\_\_$,

$f'(x) = \_\_\_$, $f'(0) = \_\_\_$,

$f''(x) = \_\_\_$, $f''(0) = \_\_\_$,

$f^{(3)}(x) = \_\_\_$, $f^{(3)}(0) = \_\_\_$, ...

$\cos(3x) = \underline{1} + \underline{0}\,x + \underline{-9/2}\,x^2 + \underline{0}\,x^3 + \ldots$

5. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.7.11.pg

Book Problem 11 $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$

Find the Taylor series for $f(x) = 7 + 8x + x^2$ centered at $a = 4$:

$f(x) = \underline{7+8x+x^2}$, $f(4) = \underline{7+32+16} = 55$

$f'(x) = \underline{8+2x}$, $f'(4) = \underline{16}$,

$f''(x) = \underline{2}$, $f''(4) = \underline{2}$,

$f^{(3)}(x) = \underline{0}$, $f^{(3)}(4) = \underline{0}$, ...

$f(x) = \underline{55} + \underline{16}(x-4) + \underline{1}(x-4)^2 + \underline{0}(x-4)^3 + \ldots$

$f(x) = f(4) + \frac{f'(4)}{1!}(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \cdots$

$= 55 + \frac{16}{1}(x-4) + \frac{2}{2}(x-4)^2 + \frac{0}{\underbrace{3!}_{=6}}(x-4)^3$

$= 55 + 16(x-4) + (x-4)^2 + 0(x-4)^3 + 0$

6. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.7.14.pg

Book Problem 14 $f(x) = f(3) + \frac{f'(3)}{1!}(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \cdots$

Find the Taylor series for $f(x) = 6\ln(6x)$ centered at $a = 3$:

$f(x) = 6\ln(6x)$, $f(3) = 6\ln 18$

$f'(x) = 6/x$, $f'(3) = 6/3 = 2$

$f''(x) = \frac{-6}{x^2}$, $f''(3) = -6/9 = -2/3$

$f^{(3)}(x) =$ ____, $f^{(3)}(3) =$ ____, ...

$6\ln(6x) = 6\ln 18 + 2\,(x-3) + \frac{-2/3}{2!} = -\frac{1}{3}\,(x-3)^2 +$ ____ $(x-3)^3 + \ldots$

$f(x) = 6\ln(6x) \Longrightarrow f'(x) = 6 \cdot \frac{6}{6x} = \frac{6}{x}$

$\Rightarrow f''(x) = \frac{-6}{x^2} = -6x^{-2}$

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7. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.7.23.pg

Book Problem 23 $(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$

Use the binomial series to expand the following functions as a power series. Give the first 3 non-zero terms.

$\frac{1}{5}\left(-\frac{4}{5}\right) = \frac{-4}{25}$

$k=\frac{1}{5}$ $f(x) = \sqrt[5]{1+x} = (1+x)^{1/5} = 1 + \frac{1}{5}x + \frac{\frac{1}{5}\left(\frac{1}{5}-1\right)}{2!}x^2 + \cdots = \boxed{1 + \frac{x}{5} - \frac{2}{25}x^2 + \cdots}$

$k=\frac{1}{2}$ $g(x) = \sqrt{1+3x} = 1 + \frac{1}{2}(3x) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(3x)^2 + \cdots = \boxed{1 + \frac{3}{2}x - \frac{9}{8}x^2 + \cdots}$

$k=-6$ $h(x) = \frac{1}{(1-x)^6} = (1+-x)^{-6} + \ldots = 1 + (-6)(-x) + \frac{-6(-7)}{2!}(-x)^2 + \cdots = 1 + 6x + 21x^2 \cdots$

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8. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.7.25.pg

Book Problem 25

$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots$

Use the binomial series to expand the following function as a power series. Give the first 3 non-zero terms.

$h(x) = \frac{1}{(4+x)^8} = \frac{1}{4^8} + \frac{-8}{4^9}x + \frac{9}{4^9}x^2 + \ldots$

$h(x) = \frac{1}{(4+x)^8} = \frac{1}{4^8} \cdot \frac{1}{\left(1+\frac{x}{4}\right)^8} = \frac{1}{4^8}\left(1+\frac{x}{4}\right)^{-8}$

$k=-8$

replace x by $\frac{x}{4}$

$= \frac{1}{4^8}\left(1 + (-8)\left(\frac{x}{4}\right) + \frac{(-8)(-9)}{2!}\left(\frac{x}{4}\right)^2 + \cdots\right)$

$= \frac{1}{4^8} - \frac{2}{4^8}x + \frac{9}{4^9}x^2 + \cdots$

9. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.7.29.pg

Book Problems 27 - 31

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

$\sin x = x - \frac{x^3}{3!} + \cdots$

$\tan^{-1} x = x - \frac{x^3}{3} +$

Use a Maclaurin series derived in this section to obtain the Maclaurin series for the given functions. Enter the first 3 non-zero terms only.

$f(x) = \cos(6x^3) = 1 - \frac{(6x^3)^2}{2!} + \frac{(6x^3)^4}{4!} - \cdots = \boxed{1 - 18x^6 + \frac{6^4}{24}x^{12} - \cdots}$

$f(x) = \sin(-\pi x) = (-\pi x) - \frac{(-\pi x)^3}{3!} + \frac{(-\pi x)^5}{5!} \cdots = \boxed{-\pi x + \frac{\pi^3 x^3}{6} - \frac{\pi^5 x^5}{120} + \cdots}$

$f(x) = x\tan^{-1}(6x) = x\left(6x - \frac{(6x)^3}{3} + \frac{(6x)^5}{5} \cdots\right) = 6x^2 - \frac{216x^4}{3} + \frac{6^5 x^6}{5} - \cdots$

$f(x) = x^3 e^{-x/2} = x^3\left(1 + \frac{(-x/2)}{1!} + \frac{(-x/2)^2}{2!} + \cdots\right) = \boxed{x^3 - \frac{x^4}{2} + \frac{x^5}{8} - \cdots}$

10. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.7.41.pg

Book Problem 41

$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$

a) Use the binomial series to give the first 3 non-zero terms of the power series for

$k = -1/2$, replace x by $-9x^2$

$\frac{1}{\sqrt{1-9x^2}} = (1-9x^2)^{-1/2} = 1 + \frac{-1}{2}(-9x^2) + \frac{(-1/2)(-3/2)}{2}(-9x^2)^2 + \cdots = \boxed{1 + \frac{9}{2}x^2 + \frac{3}{8}(81)x^4 + \cdots}$

b) Evaluate the indefinite integral $\int \frac{1}{\sqrt{1-9x^2}}dx =$ ____ +C.
(your answer should be an antiderivative not a power series)

$\int \frac{1}{\sqrt{1-9x^2}}dx = \int\left(1 + \frac{9}{2}x^2 + \frac{243}{8}x^4 + \cdots\right)dx$

$= x + \frac{9x^3}{6} + \frac{243x^5}{40} + \cdots + C$

$= \boxed{x + \frac{3x^3}{2} + \frac{243}{40}x^5 + \cdots + C}$

c) Using parts (a) and (b), give the first 3 non-zero terms of the power series for

$f(x) = \sin^{-1}(3x) = \int \frac{3}{\sqrt{1-(3x)^2}}dx = \boxed{3\left(x + \frac{3x^3}{2} + \frac{243}{40}x^5 + \cdots\right)}$

let $u = 3x$
$du = 3dx$

$\int \frac{3}{\sqrt{1-u^2}}\frac{du}{3} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$

11. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.7.43.pg

Book Problem 43

Evaluate the indefinite integral as an infinite series. Give the first 3 non-zero terms only.

$\int x\cos(x^7)dx = \int\left(x - \frac{x^{15}}{2} + \ldots\right)dx = C + \frac{x^2}{2} - \frac{x^{16}}{32} - \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$\cos(x^7) = 1 - \frac{(x^7)^2}{2!} + \cdots \Rightarrow x\cos(x^7) = x - \frac{x^{15}}{2} + \cdots$

Book Problems 51 - 53

$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

$1 + 5x - e^{5x} = \cancel{1} + 5\cancel{x} - \left(\cancel{1} + \frac{\cancel{5x}}{1!} + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + \cdots\right) = \frac{-25x^2}{2} - \frac{125x^3}{6}$

Use series to evaluate the following limits. Give only two non-zero terms in the power series expansions.

$\frac{1+5x-e^{5x}}{x^2} = \frac{\frac{-25}{2}x^2 - \frac{125x^3}{6}}{x^2} = \frac{-25}{2} - \frac{125x}{6}$

$\frac{1+5x-e^{5x}}{x^2} = -12.5 - \frac{125}{6}x + \cdots$, $\lim_{x\to 0} \frac{1+5x-e^{5x}}{x^2} = -12.5$

$\frac{4x - \tan^{-1} 4x}{x^3} = \_\_\_\_ + \ldots$, $\lim_{x\to 0} \frac{4x - \tan^{-1} 4x}{x^3} = \_\_\_$

$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\frac{\sin x - x + \frac{1}{6}x^3}{5x^5} = \frac{\frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{5x^5} = \frac{1}{5(5!)} - \frac{x^2}{5(7!)} \cdots + \ldots$, $\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{5x^5} = \frac{1}{5(5!)}$

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13. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.7.59.pg

Book Problem 59

$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $e^{3x} = \sum \frac{(3x)^n}{n!}$

MIX and MATCH

C 1. $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{2n}}{(2n)!}$

B 2. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}$

D 3. $\sum_{n=0}^{\infty} \frac{(-1)^n 3x^{2n+1}}{2n+1}$

A 4. $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

E 5. $\sum_{n=0}^{\infty} (-1)^n \frac{3x^{2n+1}}{(2n+1)!}$

A. e^{3x}
B. e^{-x^3}
C. $\cos(3x)$
D. $3\arctan(x)$
E. $3\sin(x)$

$e^{-x^3} = \sum \frac{(-x^3)^n}{n!} = \sum \frac{(-1)^n x^{3n}}{n!}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos(3x) = \sum \frac{(-1)^n (3x)^{2n}}{(2n)!}$

$3\arctan x = 3\left(x - \frac{x^3}{3} + \frac{x^5}{5} \cdots\right) = 3\sum (-1)^n \frac{x^{2n+1}}{2n+1}$

$3\sin x = 3\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right) = 3\sum (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(6)}(0)}{6!}x^6 + \cdots$

14. (1 pt) UNCC1242/EssentialCalculus-Stewart-Sec8.7.67.pg

Book Problem 67 $e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

Give the first 5 non-zero terms in the power series expansion and evaluate the derivative of the following functions:

$f(x) = e^{5x^2} = 1 + \frac{5x^2}{1!} + \frac{(5x^2)^2}{2!} + \frac{(5x^2)^3}{3!} + \frac{(5x^2)^4}{4!} + \cdots = 1 + 5x^2 + \frac{25x^4}{2} + \frac{125x^6}{6} + \frac{625x^8}{24}$ $+ \ldots$, $f^{(6)}(0) = \_\_\_\_$.

$\frac{f^{(6)}(0)}{6!} = \frac{125}{6} \Rightarrow f^{(6)}(0) = \frac{125(6!)}{6}$

$= 125(5!)$

$= 125(120)$

$= \cdots$

$g(x) = \tan^{-1} 5x = \_\_\_\_\_\_\_\_ + \ldots$, $g^{(7)}(0) = \_\_\_\_$.