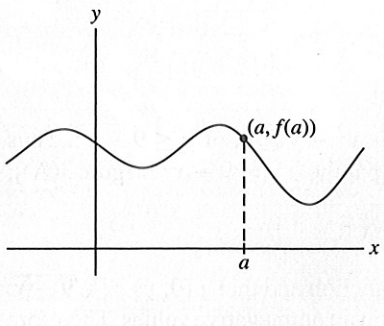


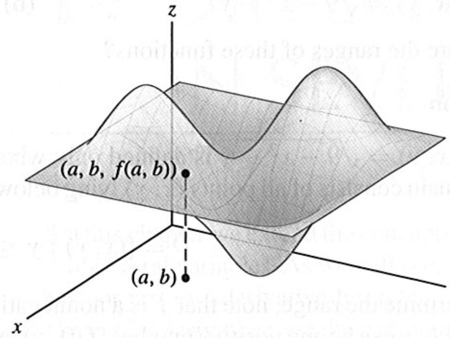


FIGURE 4

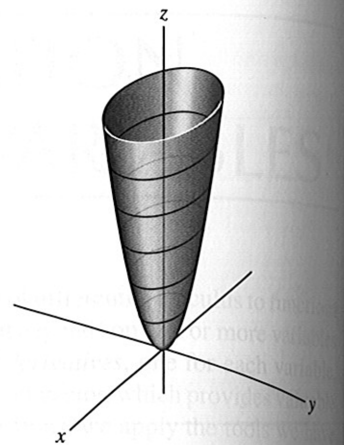


FIGURE 5 Graph of $f(x, y) = 2x^2 + 5y^2$

Plotting more complicated graphs by hand can be difficult. Fortunately, computer algebra systems eliminate the labor and greatly enhance our ability to explore functions graphically. Graphs can be rotated and viewed from different perspectives (Figure 6).

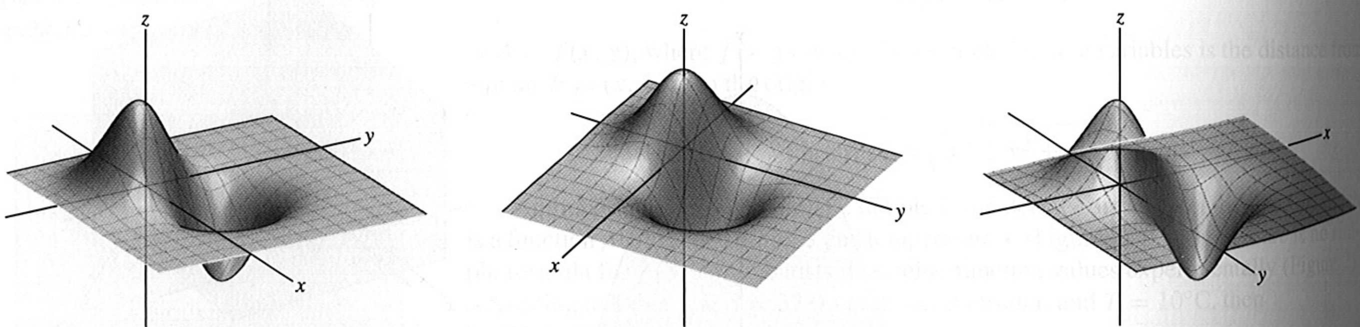


FIGURE 6 Different views of $z = e^{-x^2-y^2} - e^{-(x-1)^2-(y-1)^2}$

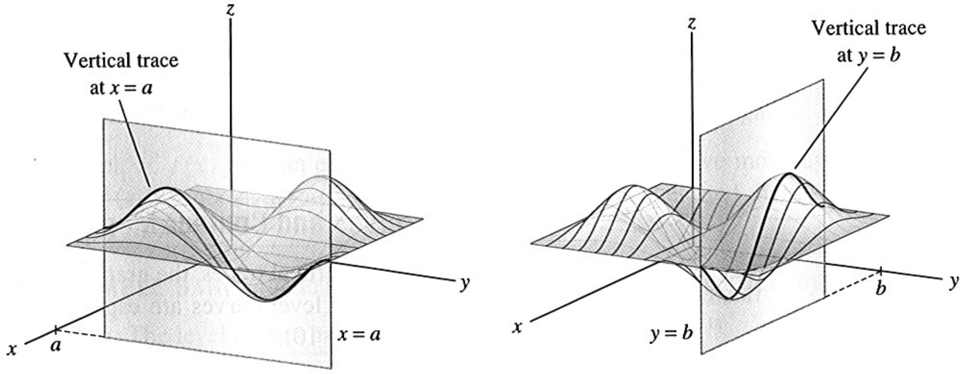
Traces and Level Curves

One way of analyzing the graph of a function $f(x, y)$ is to freeze the x-coordinate by setting $x = a$ and examine the resulting curve $z = f(a, y)$. Similarly, we may set $y = b$ and consider the curve $z = f(x, b)$. Curves of this type are called **vertical traces**. They are obtained by intersecting the graph with planes parallel to a vertical coordinate plane (Figure 7):

- **Vertical trace in the plane $x = a$:** Intersection of the graph with the vertical plane $x = a$, consisting of all points $(a, y, f(a, y))$.
- **Vertical trace in the plane $y = b$:** Intersection of the graph with the vertical plane $y = b$, consisting of all points $(x, b, f(x, b))$.

■ **EXAMPLE 3** Describe the vertical traces of $f(x, y) = x \sin y$.

Solution When we freeze the x-coordinate by setting $x = a$, we obtain the trace curve $z = a \sin y$ (see Figure 8). This is a sine curve located in the plane $x = a$. When we set $y = b$, we obtain a line $z = (\sin b)y$ of slope $\sin b$, located in the plane $y = b$. ■



(A) Vertical traces parallel to yz-plane

(B) Vertical traces parallel to xz-plane

FIGURE 7

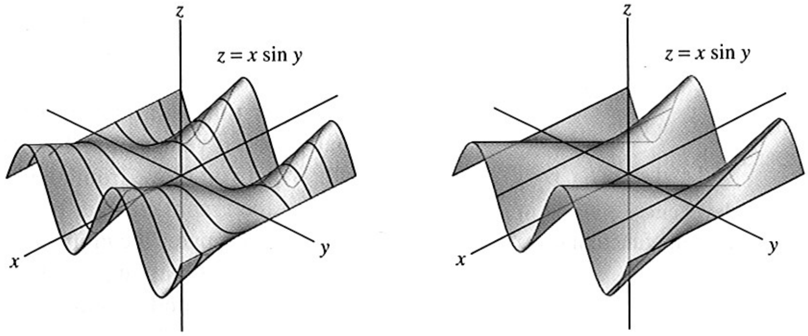


FIGURE 8 Vertical traces of $f(x, y) = x \sin y$.

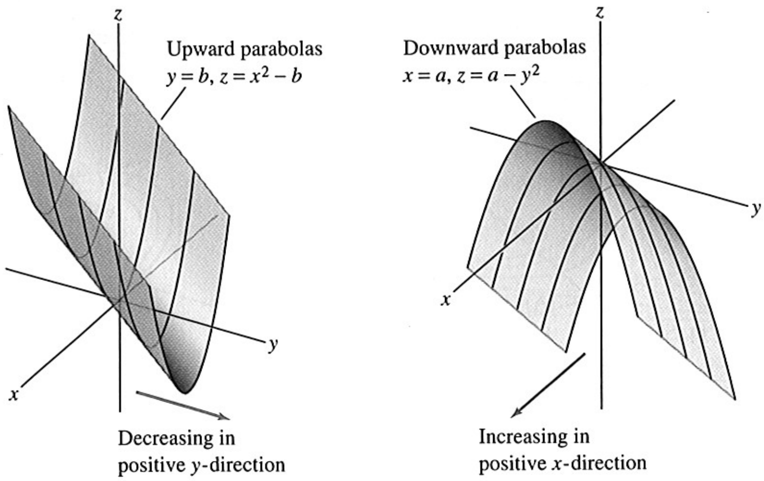
(A) The traces in the planes $x = a$ are the curves $z = a(\sin y)$.

(B) The traces in the planes $y = b$ are the lines $z = (\sin b)y$.

■ **EXAMPLE 4 Identifying Features of a Graph** Match the graphs in Figure 9 with the following functions:

(i) $f(x, y) = x - y^2$ (ii) $g(x, y) = x^2 - y$

Solution Let's compare vertical traces. The vertical trace of $f(x, y) = x - y^2$ in the plane $x = a$ is a *downward* parabola $z = a - y^2$. This matches (B). On the other hand,



(A) (B)

FIGURE 9

the vertical trace of $g(x, y)$ in the plane $y = b$ is an *upward* parabola $z = x^2 - b$. This matches (A).

Notice also that $f(x, y) = x - y^2$ is an increasing function of x (that is, $f(x, y)$ increases as x increases) as in (B), whereas $g(x, y) = x^2 - y$ is a decreasing function of y as in (A). ∎

Level Curves and Contour Maps

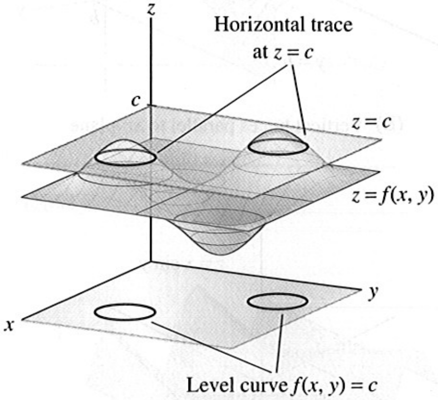


FIGURE 10 The level curve consists of all points (x, y) where the function takes on the value c.

On contour maps level curves are often referred to as ***contour lines***.

In addition to vertical traces, the graph of $f(x, y)$ has horizontal traces. These traces and their associated level curves are especially important in analyzing the behavior of the function (Figure 10):

- **Horizontal trace at height c:** Intersection of the graph with the horizontal plane $z = c$, consisting of the points $(x, y, f(x, y))$ such that $f(x, y) = c$.
- **Level curve:** The curve $f(x, y) = c$ in the xy-plane.

Thus the level curve consists of all points (x, y) in the plane where the function takes the value c. Each level curve is the projection onto the xy-plane of the horizontal trace on the graph that lies above it.

A **contour map** is a plot in the xy-plane that shows the level curves $f(x, y) = c$ for equally spaced values of c. The interval m between the values is called the **contour interval**. When you move from one level curve to next, the value of $f(x, y)$ (and hence the height of the graph) changes by $\pm m$.

Figure 11 compares the graph of a function $f(x, y)$ in (A) and its horizontal traces in (B) with the contour map in (C). The contour map in (C) has contour interval $m = 100$.

It is important to understand how the contour map indicates the steepness of the graph. If the level curves are close together, then a small move from one level curve to the next in the xy-plane leads to a large change in height. In other words, *the level curves are close together if the graph is steep* (Figure 11). Similarly, the graph is flatter when the level curves are farther apart.

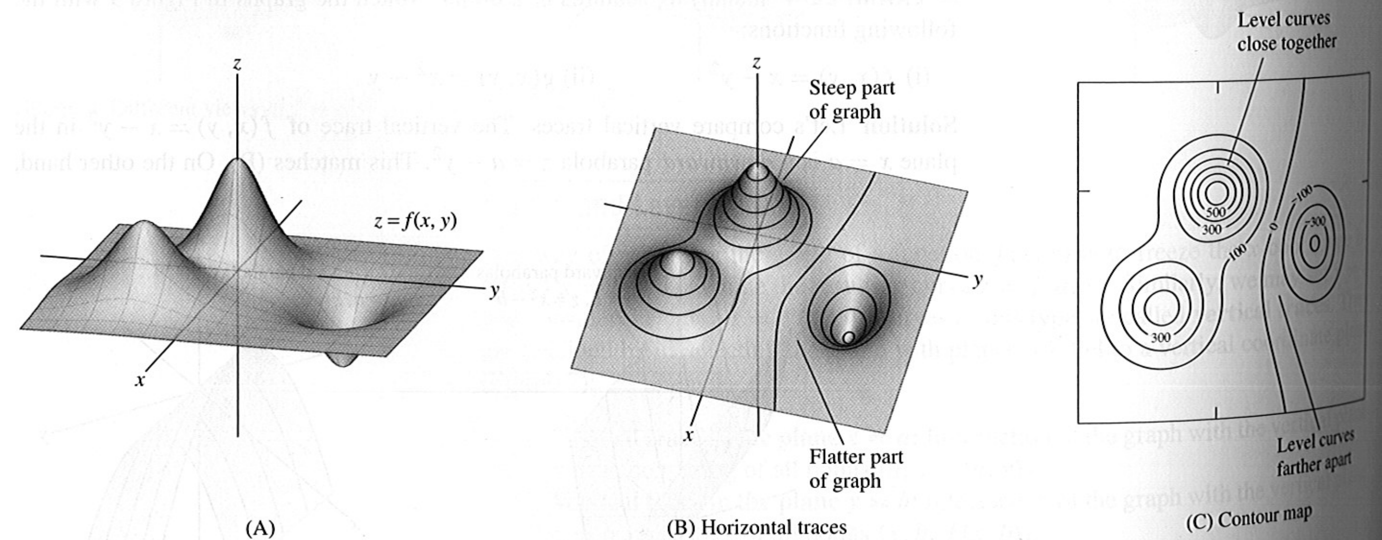


FIGURE 11

■ **EXAMPLE 5 Elliptic Paraboloid** Sketch the contour map of $f(x, y) = x^2 + 3y^2$ and comment on the spacing of the contour curves.

Solution The level curves have equation $f(x, y) = c$, or

$$x^2 + 3y^2 = c$$

- For $c > 0$, the level curve is an ellipse.
- For $c = 0$, the level curve is just the point $(0, 0)$ because $x^2 + 3y^2 = 0$ only for $(x, y) = (0, 0)$.
- The level curve is empty if $c < 0$ because $f(x, y)$ is never negative.

The graph of $f(x, y)$ is an elliptic paraboloid (Figure 12). As we move away from the origin, $f(x, y)$ increases more rapidly. The graph gets steeper, and the level curves get closer together. ■

■ **EXAMPLE 6 Hyperbolic Paraboloid** Sketch the contour map of $g(x, y) = x^2 - 3y^2$.

Solution The level curves have equation $g(x, y) = c$, or

$$x^2 - 3y^2 = c$$

- For $c \neq 0$, the level curve is the hyperbola $x^2 - 3y^2 = c$.
- For $c = 0$, the level curve consists of the two lines $x = \pm\sqrt{3}y$ because the equation $g(x, y) = 0$ factors as follows:

$$x^2 - 3y^2 = 0 = (x - \sqrt{3}y)(x + \sqrt{3}y) = 0$$

The graph of $g(x, y)$ is a hyperbolic paraboloid (Figure 13). When you stand at the origin, $g(x, y)$ increases as you move along the x-axis in either direction and decreases as you move along the y-axis in either direction. Furthermore, the graph gets steeper as you move out from the origin, so the level curves get closer together. ■

← REMINDER *The hyperbolic paraboloid in Figure 13 is often called a "saddle" or "saddle-shaped surface."*

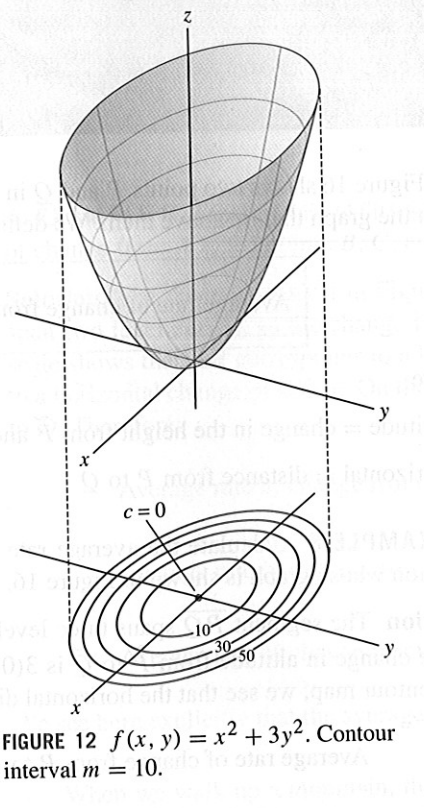


FIGURE 12 $f(x, y) = x^2 + 3y^2$. Contour interval $m = 10$.

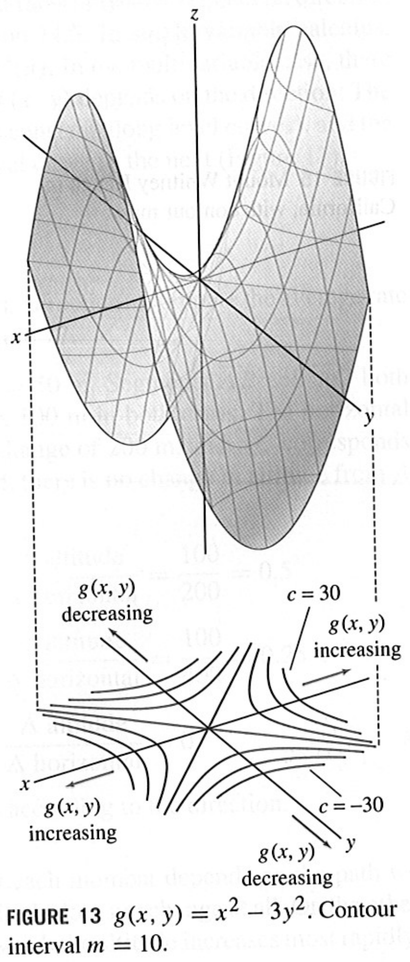


FIGURE 13 $g(x, y) = x^2 - 3y^2$. Contour interval $m = 10$.

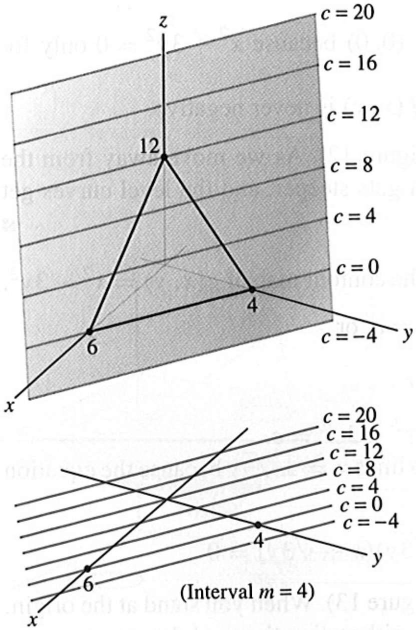


FIGURE 14 Graph and contour map of $f(x, y) = 12 - 2x - 3y$.

■ **EXAMPLE 7 Contour Map of a Linear Function** Sketch the graph of $f(x, y) = 12 - 2x - 3y$ and the associated contour map with contour interval $m = 4$.

Solution To plot the graph, which is a plane, we find the intercepts with the axes (Figure 14). The graph intercepts the z-axis at $z = f(0, 0) = 12$. To find the x-intercept, we set $y = z = 0$ to obtain $12 - 2x - 3(0) = 0$, or $x = 6$. Similarly, solving $12 - 3y = 0$ gives y-intercept $y = 4$. The graph is the plane determined by the three intercepts.

In general, the level curves of a linear function $f(x, y) = qx + ry + s$ are the lines with equation $qx + ry + s = c$. Therefore, *the contour map of a linear function consists of equally spaced parallel lines.* In our case, the level curves are the lines $12 - 2x - 3y = c$, or $2x + 3y = 12 - c$ (Figure 14). ■

How can we measure steepness quantitatively? Let's imagine the surface $z = f(x, y)$ as a mountain range. In fact, contour maps (also called topographical maps) are used extensively to describe terrain (Figure 15). We place the xy-plane at sea level, so that $f(a, b)$ is the height (also called altitude or elevation) of the mountain above sea level at the point (a, b) in the plane.

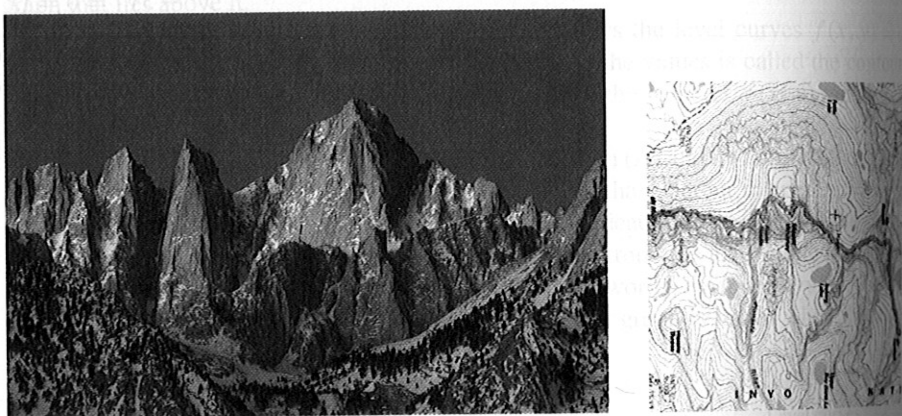


FIGURE 15 Mount Whitney Range in California, with contour map.

Figure 16 shows two points P and Q in the xy-plane, together with the points $\tilde{P}$ and $\tilde{Q}$ on the graph that lie above them. We define the **average rate of change:**

$$\text{Average rate of change from } P \text{ to } Q = \frac{\Delta \text{ altitude}}{\Delta \text{ horizontal}}$$

where

Δ altitude = change in the height from $\tilde{P}$ and $\tilde{Q}$

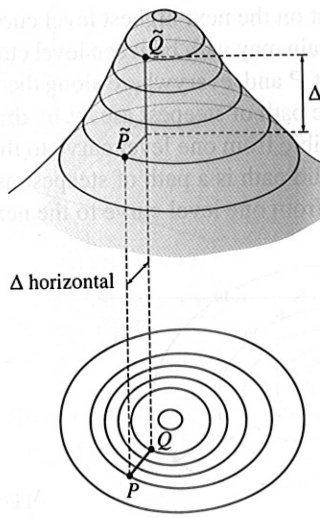
Δ horizontal = distance from P to Q

■ **EXAMPLE 8** Calculate the average rate of change of $f(x, y)$ from P to Q for the function whose graph is shown in Figure 16.

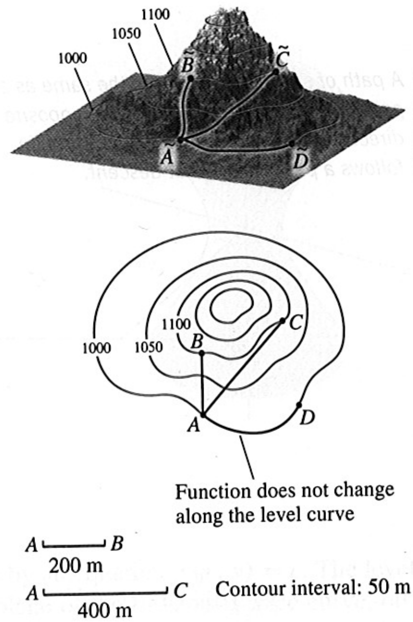
Solution The segment $\overline{PQ}$ spans three level curves and the contour interval is 0.8 km, so the change in altitude from $\tilde{P}$ to $\tilde{Q}$ is $3(0.8) = 2.4$ km. From the horizontal scale of the contour map, we see that the horizontal distance PQ is 2 km, so

$$\text{Average rate of change from } P \text{ to } Q = \frac{\Delta \text{ altitude}}{\Delta \text{ horizontal}} = \frac{2.4}{2} = 1.2$$

On average, your altitude gain is 1.2 times your horizontal distance traveled as you climb from $\tilde{P}$ to $\tilde{Q}$. ■



Contour interval: 0.8 km
Horizontal scale: 2 km
FIGURE 16



Contour interval: 50 m
FIGURE 17

CONCEPTUAL INSIGHT We will discuss the idea that rates of change depend on direction when we come to directional derivatives in Section 14.5. In single-variable calculus, we measure the rate of change by the derivative $f'(a)$. In the multivariable case, there is no single rate of change because the change in $f(x, y)$ depends on the direction: The rate is zero along a level curve (because $f(x, y)$ is constant along level curves), and the rate is nonzero in directions pointing from one level curve to the next (Figure 17).

■ **EXAMPLE 9 Average Rate of Change Depends on Direction** Compute the average rate of change from A to the points B, C, and D in Figure 17.

Solution The contour interval in Figure 17 is $m = 50$ m. Segments $\overline{AB}$ and $\overline{AC}$ both span two level curves, so the change in altitude is 100 m in both cases. The horizontal scale shows that AB corresponds to a horizontal change of 200 m, and $\overline{AC}$ corresponds to a horizontal change of 400 m. On the other hand, there is no change in altitude from A to D. Therefore:

$$\text{Average rate of change from } A \text{ to } B = \frac{\Delta \text{ altitude}}{\Delta \text{ horizontal}} = \frac{100}{200} = 0.5$$

$$\text{Average rate of change from } A \text{ to } C = \frac{\Delta \text{ altitude}}{\Delta \text{ horizontal}} = \frac{100}{400} = 0.25$$

$$\text{Average rate of change from } A \text{ to } D = \frac{\Delta \text{ altitude}}{\Delta \text{ horizontal}} = 0$$ ■

We see here explicitly that the average rate varies according to the direction.

When we walk up a mountain, the incline at each moment depends on the path we choose. If we walk "around" the mountain, our altitude does not change at all. On the other hand, at each point there is a *steepest* direction in which the altitude increases most rapidly.