Riemann Sums
Trapezoid Rule, Simpson's Rule, & Error Bounds

Riemann Sum Notation and General Information:

By Definition:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $\boxed{\Delta x = \frac{b-a}{n}}$ and x_i^* is the appropriate x-value for whichever Riemann Sums used

the common formulas for x_i^* are $\boxed{x_i = a + i \cdot \Delta x}$ and $\boxed{\overline{x_i} = \frac{x_{i-1}+x_i}{2}}$

The approximation is

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

for some value of n that you pick.

The Riemann Sum Rules:

Right Riemann Sum: $R_n = \Delta x[f(x_1) + \cdots + f(x_n)] = \sum_{i=1}^{n} f(x_i)\Delta x$

Left Riemann Sum: $L_n = \Delta x[f(x_0) + \cdots + f(x_{n-1})] = \sum_{i=1}^{n} f(x_{i-1})\Delta x$

The Midpoint Rule: $M_n = \Delta x[f(\overline{x_1}) + \cdots + f(\overline{x_n})] = \sum_{i=1}^{n} f(\overline{x_i})\Delta x$

The Trapezoid Rule: $T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$

Simpson's Rule: $S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$

where n must be even for the Simpson's Rule

The Error Bounds (ie: the maximum the absolute error can be):

The Midpoint Rule: $|E_M| \le \frac{K(b-a)^3}{24n^2}$ where $K = \max\{|f''(x)|\}$ on $a \le x \le b$

The Trapezoid Rule: $|E_T| \le \frac{K(b-a)^3}{12n^2}$ where $K = \max\{|f''(x)|\}$ on $a \le x \le b$

Simpson's Rule: $|E_S| \le \frac{K(b-a)^5}{180n^4}$ where $K = \max\{|f^{(4)}(x)|\}$ on $a \le x \le b$

Example: Consider the integral: $\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$

1. Calculate T_4

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \quad \text{and} \quad x_i = a + i \cdot \Delta x = 0 + i \cdot \frac{1}{4} = \frac{i}{4}$$

$$T_4 = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

$$= \frac{\left(\frac{1}{4}\right)}{2}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right]$$

$$= \frac{1}{8}[.39894 + 2(.38667) + 2(.35207) + 2(.30114) + .24197] = .340081844$$

2. Find the smallest n such that the approximation T_n is accurate to within .0001 of the true answer.

We need the following:

$$|E_T| \le .0001$$

$$\frac{K(b-a)^3}{12n^2} \le .0001$$

Before we go any farther, we'll find K:

$$K = \max\{|f''(x)|\} \;\; on \;\; 0 \le x \le 1$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$f'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{d}{dx}\left(-\frac{1}{2}x^2\right) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot (-x) = -\frac{x}{\sqrt{2\pi}} e^{-x^2/2}$$

$$f''(x) = -\frac{x}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{d}{dx}\left(-\frac{1}{2}x^2\right) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = \frac{(x-1)}{\sqrt{2\pi}} e^{-x^2/2}$$

on $0 \le x \le 1$, $f''(x)$ is increasing and negative, so choose $K \ge |f''(0)|$

I'll pick, $K = .4$ since $f''(0) = -.3989423$

Back to the inequality:

$$\frac{.4(1-0)^3}{12n^2} \le .0001 \quad \Rightarrow \quad \frac{.4}{12(.0001)} \le n^2$$

$$333.\overline{3} \le n^2 \quad \Rightarrow \quad 18.25741858 \le n$$

So the smallest n that will give a .0001 accuracy is $n = 19$.