1.5 EXERCISES

In Exercises 1–4, determine if the system has a nontrivial solution. Try to use as few row operations as possible.

1. $\begin{aligned} 2x_1 - 5x_2 + 8x_3 &= 0 \\ -2x_1 - 7x_2 + x_3 &= 0 \\ 4x_1 + 2x_2 + 7x_3 &= 0 \end{aligned}$

2. $\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ -2x_1 - 3x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 + 9x_3 &= 0 \end{aligned}$

3. $\begin{aligned} -3x_1 + 4x_2 - 8x_3 &= 0 \\ -2x_1 + 5x_2 + 4x_3 &= 0 \end{aligned}$

4. $\begin{aligned} 5x_1 - 3x_2 + 2x_3 &= 0 \\ -3x_1 - 4x_2 + 2x_3 &= 0 \end{aligned}$

In Exercises 5 and 6, follow the method of Examples 1 and 2 to write the solution set of the given homogeneous system in parametric vector form.

5. $\begin{aligned} 2x_1 + 2x_2 + 4x_3 &= 0 \\ -4x_1 - 4x_2 - 8x_3 &= 0 \\ -3x_2 - 3x_3 &= 0 \end{aligned}$

6. $\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + x_2 - 3x_3 &= 0 \\ -1x_1 + x_2 &= 0 \end{aligned}$

In Exercises 7–12, describe all solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form, where A is row equivalent to the given matrix.

7. $\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -6 & 6 \\ -2 & 4 & -2 \end{bmatrix}$

10. $\begin{bmatrix} -1 & -4 & 0 & -4 \\ 2 & -8 & 0 & 8 \end{bmatrix}$

11. $\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 1 & -2 & 3 & -6 & 5 & 0 \\ 0 & 0 & 0 & 1 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

13. Suppose the solution set of a certain system of linear equations can be described as $x_1 = 5 + 4x_3$, $x_2 = -2 - 7x_3$, with x_3 free. Use vectors to describe this set as a line in $\mathbb{R}^3$.

14. Suppose the solution set of a certain system of linear equations can be described as $x_1 = 5x_4$, $x_2 = 3 - 2x_4$, $x_3 = 2 + 5x_4$, with x_4 free. Use vectors to describe this set as a "line" in $\mathbb{R}^4$.

15. Describe and compare the solution sets of $x_1 + 5x_2 - 3x_3 = 0$ and $x_1 + 5x_2 - 3x_3 = -2$.

16. Describe and compare the solution sets of $x_1 - 2x_2 + 3x_3 = 0$ and $x_1 - 2x_2 + 3x_3 = 4$.

17. Follow the method of Example 3 to describe the solutions of the following system in parametric vector form. Also, give a geometric description of the solution set and compare it to that in Exercise 5.

$$\begin{aligned} 2x_1 + 2x_2 + 4x_3 &= 8 \\ -4x_1 - 4x_2 - 8x_3 &= -16 \\ -3x_2 - 3x_3 &= 12 \end{aligned}$$

18. As in Exercise 17, describe the solutions of the following system in parametric vector form, and provide a geometric comparison with the solution set in Exercise 6.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 5 \\ 2x_1 + x_2 - 3x_3 &= 13 \\ -x_1 + x_2 &= -8 \end{aligned}$$

In Exercises 19 and 20, find the parametric equation of the line through $\mathbf{a}$ parallel to $\mathbf{b}$.

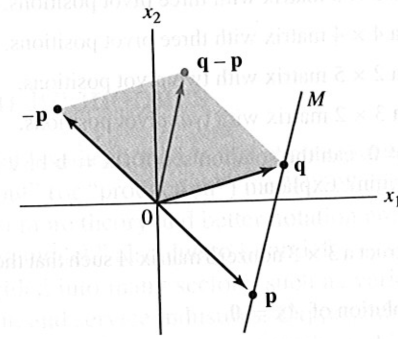
19. $\mathbf{a} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

20. $\mathbf{a} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -7 \\ 6 \end{bmatrix}$

In Exercises 21 and 22, find a parametric equation of the line M through $\mathbf{p}$ and $\mathbf{q}$. [*Hint:* M is parallel to the vector $\mathbf{q} - \mathbf{p}$. See the figure below.]

21. $\mathbf{p} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

22. $\mathbf{p} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$



In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. A homogeneous equation is always consistent.

b. The equation $A\mathbf{x} = \mathbf{0}$ gives an explicit description of its solution set.

c. The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution if and only if the equation has at least one free variable.

d. The equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ describes a line through $\mathbf{v}$ parallel to $\mathbf{p}$.

e. The solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the equation $A\mathbf{x} = \mathbf{0}$.

24. a. A homogeneous system of equations can be inconsistent.

b. If $\mathbf{x}$ is a nontrivial solution of $A\mathbf{x} = \mathbf{0}$, then every entry in $\mathbf{x}$ is nonzero.

c. The effect of adding $\mathbf{p}$ to a vector is to move the vector in a direction parallel to $\mathbf{p}$.

d. The equation $A\mathbf{x} = \mathbf{b}$ is homogeneous if the zero vector is a solution.

e. If $A\mathbf{x} = \mathbf{b}$ is consistent, then the solution set of $A\mathbf{x} = \mathbf{b}$ is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$.

25. Prove Theorem 6:

a. Suppose $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$, so that $A\mathbf{p} = \mathbf{b}$. Let $\mathbf{v}_h$ be any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$, and let $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$. Show that $\mathbf{w}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

b. Let $\mathbf{w}$ be any solution of $A\mathbf{x} = \mathbf{b}$, and define $\mathbf{v}_h = \mathbf{w} - \mathbf{p}$. Show that $\mathbf{v}_h$ is a solution of $A\mathbf{x} = \mathbf{0}$. This shows that every solution of $A\mathbf{x} = \mathbf{b}$ has the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, with $\mathbf{p}$ a particular solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v}_h$ a solution of $A\mathbf{x} = \mathbf{0}$.

26. Suppose A is the 3×3 *zero* matrix (with all zero entries). Describe the solution set of the equation $A\mathbf{x} = \mathbf{0}$.

27. Suppose $A\mathbf{x} = \mathbf{b}$ has a solution. Explain why the solution is unique precisely when $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

In Exercises 28–31, (a) does the equation $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution and (b) does the equation $A\mathbf{x} = \mathbf{b}$ have at least one solution for every possible $\mathbf{b}$?

28. A is a 3×3 matrix with three pivot positions.

29. A is a 4×4 matrix with three pivot positions.

30. A is a 2×5 matrix with two pivot positions.

31. A is a 3×2 matrix with two pivot positions.

32. If $\mathbf{b} \neq \mathbf{0}$, can the solution set of $A\mathbf{x} = \mathbf{b}$ be a plane through the origin? Explain.

33. Construct a 3×3 nonzero matrix A such that the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution of $A\mathbf{x} = \mathbf{0}$.

34. Construct a 3×3 nonzero matrix A such that the vector $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is a solution of $A\mathbf{x} = \mathbf{0}$.

35. Given $A = \begin{bmatrix} -1 & -3 \\ 7 & 21 \\ -2 & -6 \end{bmatrix}$, find one nontrivial solution of $A\mathbf{x} = \mathbf{0}$ by inspection. [*Hint*: Think of the equation $A\mathbf{x} = \mathbf{0}$ written as a vector equation.]

36. Given $A = \begin{bmatrix} 3 & -2 \\ -6 & 4 \\ 12 & -8 \end{bmatrix}$, find one nontrivial solution of $A\mathbf{x} = \mathbf{0}$ by inspection.

37. Construct a 2×2 matrix A such that the solution set of the equation $A\mathbf{x} = \mathbf{0}$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin. Then, find a vector $\mathbf{b}$ in $\mathbb{R}^2$ such that the solution set of $A\mathbf{x} = \mathbf{b}$ is *not* a line in $\mathbb{R}^2$ parallel to the solution set of $A\mathbf{x} = \mathbf{0}$. Why does this *not* contradict Theorem 6?

38. Let A be an $m \times n$ matrix and let $\mathbf{w}$ be a vector in $\mathbb{R}^n$ that satisfies the equation $A\mathbf{x} = \mathbf{0}$. Show that for any scalar c, the vector $c\mathbf{w}$ also satisfies $A\mathbf{x} = \mathbf{0}$. [That is, show that $A(c\mathbf{w}) = \mathbf{0}$.]

39. Let A be an $m \times n$ matrix, and let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ with the property that $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$. Explain why $A(\mathbf{v} + \mathbf{w})$ must be the zero vector. Then explain why $A(c\mathbf{v} + d\mathbf{w}) = \mathbf{0}$ for each pair of scalars c and d.

40. Suppose A is a 3×3 matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{b}$ does *not* have a solution. Does there exist a vector $\mathbf{y}$ in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{y}$ has a unique solution? Discuss.

SOLUTIONS TO PRACTICE PROBLEMS

1. Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -5 & 0 \\ 0 & -9 & 18 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad + 3x_3 &= 4 \\ x_2 - 2x_3 &= -1 \end{aligned}$$

Thus $x_1 = 4 - 3x_3$, $x_2 = -1 + 2x_3$, with x_3 free. The general solution in parametric vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 - 3x_3 \\ -1 + 2x_3 \\ x_3 \end{bmatrix} = \underset{\substack{\uparrow \\ \mathbf{p}}}{\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}} + x_3 \underset{\substack{\uparrow \\ \mathbf{v}}}{\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}}$$

The intersection of the two planes is the line through $\mathbf{p}$ in the direction of $\mathbf{v}$.