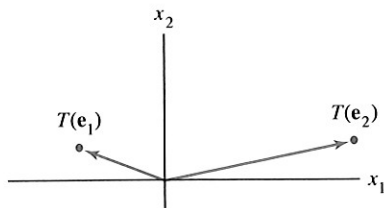


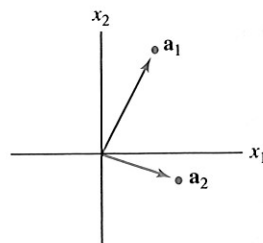
## 1.9 EXERCISES

In Exercises 1–10, assume that  $T$  is a linear transformation. Find the standard matrix of  $T$ .

1.  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^4$ ,  $T(\mathbf{e}_1) = (3, 1, 3, 1)$ , and  $T(\mathbf{e}_2) = (-5, 2, 0, 0)$ , where  $\mathbf{e}_1 = (1, 0)$  and  $\mathbf{e}_2 = (0, 1)$ .
2.  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ ,  $T(\mathbf{e}_1) = (1, 4)$ ,  $T(\mathbf{e}_2) = (-2, 9)$ , and  $T(\mathbf{e}_3) = (3, -8)$ , where  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$  are the columns of the  $3 \times 3$  identity matrix.
3.  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a vertical shear transformation that maps  $\mathbf{e}_1$  into  $\mathbf{e}_1 - 3\mathbf{e}_2$ , but leaves  $\mathbf{e}_2$  unchanged.
4.  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a horizontal shear transformation that leaves  $\mathbf{e}_1$  unchanged and maps  $\mathbf{e}_2$  into  $\mathbf{e}_2 + 2\mathbf{e}_1$ .
5.  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  rotates points (about the origin) through  $\pi/2$  radians (counterclockwise).
6.  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  rotates points (about the origin) through  $-3\pi/2$  radians (clockwise).
7.  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  first rotates points through  $-3\pi/4$  radians (clockwise) and then reflects points through the horizontal  $x_1$ -axis. [Hint:  $T(\mathbf{e}_1) = (-1/\sqrt{2}, 1/\sqrt{2})$ .]
8.  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  first performs a horizontal shear that transforms  $\mathbf{e}_2$  into  $\mathbf{e}_2 + 2\mathbf{e}_1$  (leaving  $\mathbf{e}_1$  unchanged) and then reflects points through the line  $x_2 = -x_1$ .
9.  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  first reflects points through the horizontal  $x_1$ -axis and then rotates points  $-\pi/2$  radians.
10.  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  first reflects points through the horizontal  $x_1$ -axis and then reflects points through the line  $x_2 = x_1$ .
11. A linear transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  first reflects points through the  $x_1$ -axis and then reflects points through the  $x_2$ -axis. Show that  $T$  can also be described as a linear transformation that rotates points about the origin. What is the angle of that rotation?
12. Show that the transformation in Exercise 10 is merely a rotation about the origin. What is the angle of the rotation?
13. Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear transformation such that  $T(\mathbf{e}_1)$  and  $T(\mathbf{e}_2)$  are the vectors shown in the figure. Using the figure, sketch the vector  $T(2, 1)$ .



14. Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a linear transformation with standard matrix  $A = [\mathbf{a}_1 \ \mathbf{a}_2]$ , where  $\mathbf{a}_1$  and  $\mathbf{a}_2$  are shown in the figure at the top of column 2. Using the figure, draw the image of  $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$  under the transformation  $T$ .



In Exercises 15 and 16, fill in the missing entries of the matrix, assuming that the equation holds for all values of the variables.

$$15. \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 4x_2 \\ x_1 - x_3 \\ -x_2 + 3x_3 \end{bmatrix}$$

$$16. \begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_2 \\ x_1 + 4x_2 \\ x_2 \end{bmatrix}$$

In Exercises 17–20, show that  $T$  is a linear transformation by finding a matrix that implements the mapping. Note that  $x_1, x_2, \dots$  are not vectors but are entries in vectors.

17.  $T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2, 0, 2x_2 + x_4, x_2 - x_4)$
18.  $T(x_1, x_2) = (x_1 + 4x_2, 0, x_1 - 3x_2, x_1)$
19.  $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$
20.  $T(x_1, x_2, x_3, x_4) = 3x_1 + 4x_3 - 2x_4$  (Notice:  $T: \mathbb{R}^4 \rightarrow \mathbb{R}$ )
21. Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a linear transformation such that  $T(x_1, x_2) = (x_1 + x_2, 4x_1 + 5x_2)$ . Find  $\mathbf{x}$  such that  $T(\mathbf{x}) = (3, 8)$ .
22. Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be a linear transformation with  $T(x_1, x_2) = (2x_1 - x_2, -3x_1 + x_2, 2x_1 - 3x_2)$ . Find  $\mathbf{x}$  such that  $T(\mathbf{x}) = (0, -1, -4)$ .

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. A linear transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is completely determined by its effect on the columns of the  $n \times n$  identity matrix.  
b. If  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  rotates vectors about the origin through an angle  $\varphi$ , then  $T$  is a linear transformation.  
c. When two linear transformations are performed one after another, the combined effect may not always be a linear transformation.  
d. A mapping  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is onto  $\mathbb{R}^m$  if every vector  $\mathbf{x}$  in  $\mathbb{R}^n$  maps onto some vector in  $\mathbb{R}^m$ .  
e. If  $A$  is a  $3 \times 2$  matrix, then the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  cannot be one-to-one.
24. a. If  $A$  is a  $4 \times 3$  matrix, then the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  maps  $\mathbb{R}^3$  onto  $\mathbb{R}^4$ .

- b. Every linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  is a matrix transformation.
- c. The columns of the standard matrix for a linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  are the images of the columns of the  $n \times n$  identity matrix under  $T$ .
- d. A mapping  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is one-to-one if each vector in  $\mathbb{R}^n$  maps onto a unique vector in  $\mathbb{R}^m$ .
- e. The standard matrix of a horizontal shear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  has the form  $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ , where  $a$  and  $d$  are  $\pm 1$ .

In Exercises 25–28, determine if the specified linear transformation is (a) one-to-one and (b) onto. Justify each answer.

25. The transformation in Exercise 17

26. The transformation in Exercise 2

27. The transformation in Exercise 19

28. The transformation in Exercise 14

In Exercises 29 and 30, describe the possible echelon forms of the standard matrix for a linear transformation  $T$ . Use the notation of Example 1 in Section 1.2.

29.  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$  is one-to-one.      30.  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  is onto.

31. Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation, with  $A$  its standard matrix. Complete the following statement to make it true: “ $T$  is one-to-one if and only if  $A$  has \_\_\_\_\_ pivot columns.” Explain why the statement is true. [Hint: Look in the exercises for Section 1.7.]

32. Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation, with  $A$  its standard matrix. Complete the following statement to make it true: “ $T$  maps  $\mathbb{R}^n$  onto  $\mathbb{R}^m$  if and only if  $A$  has \_\_\_\_\_ pivot columns.” Find some theorems that explain why the statement is true.

33. Verify the uniqueness of  $A$  in Theorem 10. Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation such that  $T(\mathbf{x}) = B\mathbf{x}$  for some  $m \times n$  matrix  $B$ . Show that if  $A$  is the standard matrix for  $T$ , then  $A = B$ . [Hint: Show that  $A$  and  $B$  have the same columns.]

34. Let  $S : \mathbb{R}^p \rightarrow \mathbb{R}^n$  and  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be linear transformations. Show that the mapping  $\mathbf{x} \mapsto T(S(\mathbf{x}))$  is a linear transformation (from  $\mathbb{R}^p$  to  $\mathbb{R}^m$ ). [Hint: Compute  $T(S(c\mathbf{u} + d\mathbf{v}))$  for  $\mathbf{u}, \mathbf{v}$  in  $\mathbb{R}^p$  and scalars  $c$  and  $d$ . Justify each step of the computation, and explain why this computation gives the desired conclusion.]

35. If a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  maps  $\mathbb{R}^n$  onto  $\mathbb{R}^m$ , can you give a relation between  $m$  and  $n$ ? If  $T$  is one-to-one, what can you say about  $m$  and  $n$ ?

36. Why is the question “Is the linear transformation  $T$  onto?” an existence question?

[M] In Exercises 37–40, let  $T$  be the linear transformation whose standard matrix is given. In Exercises 37 and 38, decide if  $T$  is a one-to-one mapping. In Exercises 39 and 40, decide if  $T$  maps  $\mathbb{R}^5$  onto  $\mathbb{R}^5$ . Justify your answers.

37. 
$$\begin{bmatrix} -5 & 6 & -5 & -6 \\ 8 & 3 & -3 & 8 \\ 2 & 9 & 5 & -12 \\ -3 & 2 & 7 & -12 \end{bmatrix}$$

38. 
$$\begin{bmatrix} 7 & 5 & 9 & -9 \\ 5 & 6 & 4 & -4 \\ 4 & 8 & 0 & 7 \\ -6 & -6 & 6 & 5 \end{bmatrix}$$

39. 
$$\begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 6 & -8 & 5 & 12 & -8 \\ -7 & 10 & -8 & -9 & 14 \\ 3 & -5 & 4 & 2 & -6 \\ -5 & 6 & -6 & -7 & 3 \end{bmatrix}$$

40. 
$$\begin{bmatrix} 9 & 43 & 5 & 6 & -1 \\ 14 & 15 & -7 & -5 & 4 \\ -8 & -6 & 12 & -5 & -9 \\ -5 & -6 & -4 & 9 & 8 \\ 13 & 14 & 15 & 3 & 11 \end{bmatrix}$$

### SOLUTION TO PRACTICE PROBLEM

WEB

Follow what happens to  $\mathbf{e}_1$  and  $\mathbf{e}_2$ . See Fig. 5. First,  $\mathbf{e}_1$  is unaffected by the shear and then is reflected into  $-\mathbf{e}_1$ . So  $T(\mathbf{e}_1) = -\mathbf{e}_1$ . Second,  $\mathbf{e}_2$  goes to  $\mathbf{e}_2 - .5\mathbf{e}_1$  by the shear

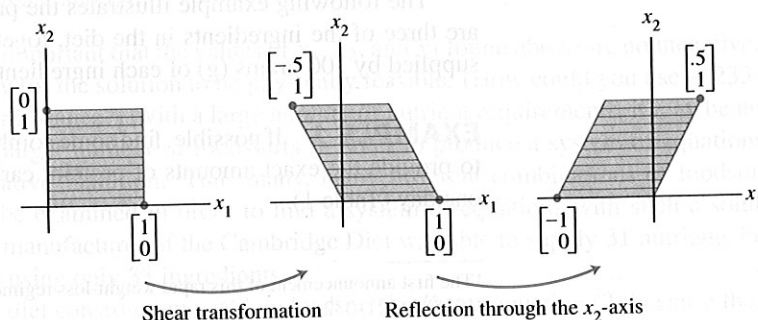


FIGURE 5 The composition of two transformations.