34. $\det EA = \begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = akd - bkc = k(ad - bc)$

$= (\det E)(\det A)$

36. $\det EA = \begin{vmatrix} a & b \\ ka + c & kb + d \end{vmatrix} = a(kb + d) - b(ka + c)$

$= akb + ad - bka - bc = (+1)(ad - bc)$

$= (\det E)(\det A)$

38. $\det kA = k^2 \cdot \det A$

39. **a.** True. See the paragraph preceding the definition of $\det A$.

b. False. See the definition of cofactor, preceding Theorem 1.

40. **a.** False. See Theorem 1.

b. False. See Theorem 2.

42. The area of the parallelogram and the determinant of $[\mathbf{v}\ \ \mathbf{u}]$ are both bc. The determinant of $[\mathbf{u}\ \ \mathbf{v}]$ is $-bc$. Both matrices determine the same parallelogram, with base of length c and height b.

44. **[M]** Theorem 6 in Section 3.2 will show that $\det AB = (\det A)(\det B)$.

46. **[M]** If A is invertible, then $\det A \neq 0$, by Theorem 4 in Section 3.2. Students will be asked in Exercise 31 of Section 3.2 to prove that $\det A^{-1} = 1/(\det A)$.

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2. A constant may be factored out of one row.

4. A row replacement operation does not change the determinant.

6. -18 **8.** 0 **10.** 24 **12.** 114

14. 0 **16.** 21 **18.** 7 **20.** 7

22. Not invertible **24.** Linearly independent

26. Linearly dependent

27. **a.** True. Theorem 3(a).

b. False. If scaling operations are used to produce U, then the formula described may not give $\det A$. See the paragraph following Example 2.

c. True. See the remark following Theorem 4.

d. False. See the warning after Example 5.

28. **a.** True. By Theorem 3(b), the first interchange changes only the sign of the determinant, so the second interchange restores the original sign of the determinant.

b. False. True when A is triangular (Theorem 2 in Section 3.1).

c. False. The conditions described provide only some cases when $\det A$ is zero. See the paragraph after Theorem 4.

d. False. See Theorem 5.

30. If two rows are equal, interchange them. This doesn't change the matrix, but the sign of the determinant is reversed. This is possible only if the determinant is zero. The result about columns can be explained the same way, or one can remark that if A has two equal columns, then A^T has two equal rows. In this case, $\det A^T = 0$. So $\det A = 0$, too, by Theorem 5.

32. $\det(rA) = r^n \cdot \det A$

34. $\det(PAP^{-1}) = (\det P)(\det A)(\det P^{-1})$ By Theorem 6

$= (\det P)(\det A)(\det P)^{-1}$ By Exercise 31

$= \det A$

36. $0 = \det A^4 = (\det A)^4$, by Theorem 6. So $\det A = 0$, which implies that A is not invertible, by Theorem 4.

38. $\det AB = \det \begin{bmatrix} 6 & 0 \\ -2 & 0 \end{bmatrix} = 0$

$(\det A)(\det B) = (-6 + 6)(-4 + 2) = 0$

40. **a.** -2

b. 32

c. -16

d. 1

e. -1

42. $\det(A + B) = \det \begin{bmatrix} 1 + a & b \\ c & 1 + d \end{bmatrix} =$

$1 + a + d + ad - bc$. Also $\det A + \det B = 1 + (ad - bc)$. Since $\det(A + B) - (\det A + \det B) = a + d$, we have $\det(A + B) = \det A + \det B$ if and only if $a + d = 0$.

44. $\det AE = \det(AE)^T$ Theorem 5

$= \det E^T A^T$ Section 2.1

$= (\det E^T)(\det A^T)$ Theorem 6

$= (\det E)(\det A)$ Theorem 5 used twice

46. **[M]** For A as in Exercise 9 of Section 2.3, $\det A = 1$ and $\text{cond } A = 23683$. Although A is nearly singular, it has an inverse:

$$A^{-1} = \begin{bmatrix} -19 & -14 & 0 & 7 \\ -549 & -401 & -2 & 196 \\ 267 & 195 & 1 & -95 \\ -278 & -203 & -1 & 99 \end{bmatrix}$$

The determinant is sensitive to scaling, but the condition number does not change:

$\det(10A) = 10^4(-1)$, $\det(0.1A) = 10^{-4}(-1)$, but

$\text{cond}(10A) = \text{cond}(0.1A) = \text{cond}\,A$

The same things happen when $A = I_4$.

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2. $\begin{bmatrix} 5/3 \\ -2/3 \end{bmatrix}$ **4.** $\begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}$ **6.** $\begin{bmatrix} -4 \\ 13 \\ -1 \end{bmatrix}$

8. All real s; $x_1 = \dfrac{3s+2}{3(s^2+3)}$, $x_2 = \dfrac{2s-9}{5(s^2+3)}$

10. $s \neq 0, 1/4$; $x_1 = \dfrac{6s-2}{3s(4s-1)}$, $x_2 = \dfrac{1}{3(4s-1)}$

12. $\text{adj}\,A = \begin{bmatrix} -1 & 3 & 7 \\ 0 & 0 & 5 \\ 2 & -1 & -4 \end{bmatrix}$, $A^{-1} = \dfrac{1}{5}\begin{bmatrix} -1 & 3 & 7 \\ 0 & 0 & 5 \\ 2 & -1 & -4 \end{bmatrix}$

14. $\text{adj}\,A = \begin{bmatrix} 5 & -3 & -8 \\ 2 & -2 & -3 \\ -4 & 3 & 6 \end{bmatrix}$, $A^{-1} = (-1)\begin{bmatrix} 5 & -3 & -8 \\ 2 & -2 & -3 \\ -4 & 3 & 6 \end{bmatrix}$

16. $\text{adj}\,A = \begin{bmatrix} -9 & -6 & 14 \\ 0 & 3 & -1 \\ 0 & 0 & -3 \end{bmatrix}$,

$$A^{-1} = -\frac{1}{9}\begin{bmatrix} -9 & -6 & 14 \\ 0 & 3 & -1 \\ 0 & 0 & -3 \end{bmatrix}$$

18. Each cofactor in A is an integer because it is just a sum of products of entries of A. Hence all the entries in adj A are integers. Since $\det A = 1$, the inverse formula in Theorem 8 shows that all the entries in A^{-1} are integers.

20. 7 **22.** 21 **24.** 15

26. By definition, $\mathbf{p} + S$ is the set of all vectors of the form $\mathbf{p} + \mathbf{v}$, where $\mathbf{v}$ is in S. Applying T to a typical vector in $\mathbf{p} + S$, we have $T(\mathbf{p} + \mathbf{v}) = T(\mathbf{p}) + T(\mathbf{v})$. This vector is in $\mathbf{p} + S$, we have $T(\mathbf{p}) + T(S)$. This proves that T maps the set denoted by $T(\mathbf{p}) + T(S)$. This proves that T maps the set $\mathbf{p} + S$ into the set $T(\mathbf{p}) + T(S)$.

Conversely, any vector in $T(\mathbf{p}) + T(S)$ has the form $T(\mathbf{p}) + T(\mathbf{v})$ for some $\mathbf{v}$ in S. This vector may be written as $T(\mathbf{p} + \mathbf{v})$. This shows that every vector in $T(\mathbf{p}) + T(S)$ is the image under T of some point in $\mathbf{p} + S$.

28. Use Theorem 10. Or, compute the vectors that determine the image, namely, the columns of

$\begin{bmatrix} 14 & 2 \end{bmatrix}$

30. Let $\mathbf{p} = (x_3, y_3)$ and let $R' = R - \mathbf{p}$. The vertices of R' are $\mathbf{v} = (x_1 - x_3, y_1 - y_3)$, $\mathbf{v}_2 = (x_2 - x_3, y_2 - y_3)$, and the origin. Then

$$\{\text{area of } R\} = \{\text{area of } R'\}$$
$$= \frac{1}{2}\left\{\begin{array}{c}\text{area of parallelogram} \\ \text{determined by } \mathbf{v}_1 \text{ and } \mathbf{v}_2\end{array}\right\}$$
$$= \frac{1}{2}\left|\det\begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix}\right| \qquad (1)$$

Also, using row operations, we get

$$\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = \det\begin{bmatrix} x_1 - x_3 & y_1 - y_3 & 0 \\ x_2 - x_3 & y_2 - y_3 & 0 \\ x_3 & y_3 & 1 \end{bmatrix}$$
$$= \det\begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}$$
$$= \det\begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix}$$

This calculation and (1) give the desired result.

32. From the formula in the exercise,

$\{\text{volume of } S\} = \frac{1}{3}\{\text{area of base}\}\cdot\{\text{height}\} = \frac{1}{6}$

because the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ have unit length. The tetrahedron S' with vertices at $\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ is the image of S under the linear transformation T such that $T(\mathbf{e}_1) = \mathbf{v}_1$, $T(\mathbf{e}_2) = \mathbf{v}_2$, and $T(\mathbf{e}_3) = \mathbf{v}_3$. The standard matrix for T is $A = [\,\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3\,]$. By Theorem 10,

$\{\text{volume of } S'\} = |\det A|\cdot\frac{1}{6} = \frac{1}{6}|\det[\,\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3\,]|$

34. **[M]** MATLAB:

```
x2 = det([A(:,1)  b  A(:,3:4)])/det(A)
```

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1. a. True. The columns of A are linearly dependent.

b. True. See Exercise 30 in Section 3.2.

c. False. See Theorem 3(c); in this case $\det 5A = 5^3 \det A$.

d. False. Consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, and $A + B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$.

e. False. By Theorem 6, $\det A^3 = 2^3$.

f. False. See Theorem 3(b).

g. True. See Theorem 3(c).

h. True. See Theorem 3(a).

i. False. See Theorem 5.

j. False. See Theorem 3(c); this statement is false for $n \times n$ invertible matrices with n an even integer.

k. True. See Theorems 6 and 5; $\det A^T A = (\det A)^2$.

$$\begin{bmatrix} a-b & -a+b & 0 & \cdots & 0 \\ 0 & a-b & -a+b & & 0 \\ 0 & 0 & a-b & & 0 \\ \vdots & & & \ddots & \vdots \\ b & b & b & \cdots & a \end{bmatrix}$$

b. Since column replacement operations are equivalent to row operations on A^T and $\det A^T = \det A$, column replacement operations do not change the determinant of the matrix. The resulting matrix is

$$\begin{bmatrix} a-b & 0 & 0 & \cdots & 0 \\ 0 & a-b & 0 & & 0 \\ 0 & 0 & a-b & & 0 \\ \vdots & & & \ddots & \vdots \\ b & 2b & 3b & \cdots & a+(n-1)b \end{bmatrix}$$

c. Since the preceding matrix is a lower triangular matrix with the same determinant as A,

$$\det A = (a-b)^{n-1}(a+(n-1)b)$$

18. **[M]** **a.** $(3-8)^3[3+(3)8] = -3375$

b. $(8-3)^4[8+(4)3] = 12{,}500$

20. **[M]** Compute:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 6 \end{vmatrix} = 6, \quad \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 6 & 6 \\ 1 & 3 & 6 & 9 \end{vmatrix} = 18,$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 & 3 \\ 1 & 3 & 6 & 6 & 6 \\ 1 & 3 & 6 & 9 & 9 \\ 1 & 3 & 6 & 9 & 12 \end{vmatrix} = 54 = 18 \cdot 3$$

Conjecture:

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 3 & & 3 \\ 1 & 3 & 6 & & 6 \\ \vdots & & & \ddots & \vdots \\ 1 & 3 & 6 & \cdots & 3(n-1) \end{vmatrix} = 2 \cdot 3^{n-2}$$

To confirm the conjecture, use row replacement operations to create zeros below the first pivot and then below the second pivot. The resulting matrix is

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 2 & 2 & 2 & 2 & & 2 \\ 0 & 0 & 3 & 3 & 3 & 3 & & 3 \\ 0 & 0 & 3 & 6 & 6 & 6 & & 6 \\ 0 & 0 & 3 & 6 & 9 & 9 & & 9 \\ \vdots & & & & & \ddots & & \vdots \\ 0 & 0 & 3 & 6 & 9 & 12 & \cdots & 3(n-2) \end{vmatrix}$$

This matrix has the same determinant as the original matrix, and is recognizable as a block matrix of the form

$$\begin{bmatrix} A & B \\ O & D \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix},$$

$$D = \begin{bmatrix} 3 & 3 & 3 & 3 & \cdots & 3 \\ 3 & 6 & 6 & 6 & & 6 \\ 3 & 6 & 9 & 9 & & 9 \\ \vdots & & & & \ddots & \vdots \\ 3 & 6 & 9 & 12 & \cdots & 3(n-2) \end{bmatrix} = 3\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & 2 & & 2 \\ 1 & 2 & 3 & 3 & & 3 \\ \vdots & & & & \ddots & \vdots \\ 1 & 2 & 3 & 4 & \cdots & n-2 \end{bmatrix}$$

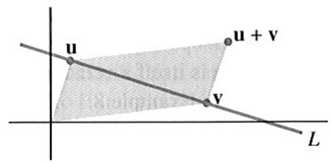
Use Exercise 14(c) to find that the determinant of the matrix $\begin{bmatrix} A & B \\ O & D \end{bmatrix}$ is $(\det A)(\det D) = 2 \det D$, and then use Exercise 32 in Section 3.2 and Exercise 19 above to show that $\det D = 3^{n-2}$.

CHAPTER 4

Section 4.1, page 223

2. **a.** Given $\begin{bmatrix} x \\ y \end{bmatrix}$ in W and any scalar c, the vector $c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ is in W because $(cx)(cy) = c^2(xy) \geq 0$, since $xy \geq 0$.

b. *Example:* If $\mathbf{u} = \begin{bmatrix} -1 \\ -7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then $\mathbf{u}$ and $\mathbf{v}$ are in W, but $\mathbf{u} + \mathbf{v}$ is not in W.

4. 

u and v are on the line, but u + v is not.

6. No, the zero polynomial is not in the set.

8. Yes. The zero vector is in the set, H. If $\mathbf{p}$ and $\mathbf{q}$ are in H, then $(\mathbf{p}+\mathbf{q})(0) = \mathbf{p}(0) + \mathbf{q}(0) = 0$, so $\mathbf{p}+\mathbf{q}$ is in H. Also, for any scalar c, $(c\mathbf{p})(0) = c\cdot\mathbf{p}(0) = c\cdot 0 = 0$, so $c\mathbf{p}$ is in H.

10. $H = \text{Span}\{\mathbf{v}\}$, where $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$. By Theorem 1, H is a subspace of $\mathbb{R}^3$.

12. $W = \text{Span}\{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}$. By Theorem 1, W is a subspace of $\mathbb{R}^4$.

14. No, because the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{w}$ has no solution, as revealed by an echelon form of the augmented matrix for this equation.

16. Not a vector space because the zero vector is not in W

18. $S = \left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

20. **a.** The constant function $\mathbf{f}(t) = 0$ is continuous. The sum of two continuous functions is continuous. A constant multiple of a continuous function is continuous.

22. Yes. See the proof of Theorem 12 in Section 2.8 for a proof that is similar to the one needed here.

23. **a.** False. The zero vector in V is the function $\mathbf{f}$ whose values $\mathbf{f}(t)$ are zero *for all* t in $\mathbb{R}$. See Example 5.

b. False. See the definition of a vector. An arrow in three-dimensional space is an example of a vector, but not every vector is such an arrow.

c. False. Exercises 1, 2, and 3 each provide an example of a subset that contains the zero vector but is not a subspace.

d. True. See the paragraph before Example 6.

e. False. Digital signals are used. See Example 3.

24. **a.** True. See the definition of a vector space.

b. True. See statement (3) in the box before Example 1.

c. True. See the paragraph before Example 6.

d. False. See Example 8.

e. False. The second and third parts of the conditions are stated incorrectly. In part (ii) here, for example, there is no statement that $\mathbf{u}$ and $\mathbf{v}$ represent all possible elements of H.

26. **a.** 3 **b.** 5 **c.** 4

28. **a.** 4 **b.** 7 **c.** 3 **d.** 5 **e.** 4

30.
$$\begin{aligned} \mathbf{u} &= 1\cdot\mathbf{u} && \text{Axiom 10} \\ &= c^{-1}c\cdot\mathbf{u} = c^{-1}(c\mathbf{u}) && \text{Axiom 9} \\ &= c^{-1}\mathbf{0} = \mathbf{0} && \text{Property (2)} \end{aligned}$$

32. Both H and K contain the zero vector of V because they are subspaces of V. Hence $\mathbf{0}$ is in $H \cap K$. Take $\mathbf{u}$ and $\mathbf{v}$ in $H \cap K$. Then $\mathbf{u}$ and $\mathbf{v}$ are in both H and K. Since H is a subspace, $\mathbf{u}+\mathbf{v}$ is in H. Likewise, $\mathbf{u}+\mathbf{v}$ is in K. Hence $\mathbf{u}+\mathbf{v}$ is in $H \cap K$. For any scalar c, the vector $c\mathbf{u}$ is in both H and K because they are subspaces. Hence $c\mathbf{u}$ is in $H \cap K$. Thus $H \cap K$ is a subspace.

The union of two subspaces is not, in general, a subspace. In $\mathbb{R}^2$, let H be the x-axis and K the y-axis. The sum of a nonzero vector in H and a nonzero vector in K is not on either the x-axis or the y-axis. So $H \cup K$ is not closed under vector addition, and $H \cup K$ is not a subspace of $\mathbb{R}^2$.

34. A proof that $H + K = \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$ has two parts. First, one must show that $H + K$ is a subset of $\text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$. Second, one must show that $\text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$ is a subset of $H + K$.

(1) A typical vector H has the form $c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$ and a typical vector in K has the form $d_1\mathbf{v}_1 + \cdots + d_q\mathbf{v}_q$. The sum of these two vectors is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q$ and so belongs to $\text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$. Thus $H + K$ is a subset of $\text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$.

(2) Each of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q$ belongs to $H + K$, by Exercise 33(b), and so any linear combination of these vectors belongs to $H + K$, since $H + K$ is a subspace, by Exercise 33(a). Thus, $\text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$ is a subset of $H + K$.

36. **[M]** An echelon form of $[A \;\; \mathbf{y}]$ shows that $A\mathbf{x} = \mathbf{y}$ is consistent. In fact, $\mathbf{x} = (5.5, -2, 3.5)$.

38. **[M]** The functions are $\sin 3t$, $\cos 4t$, and $\sin 5t$.

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2. $\begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so **w** is in Nul A.

4. $\begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ **6.** $\begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

8. W is not a subspace because **0** is not in W. The vector (0, 0, 0) does not satisfy the condition $5r - 1 = s + 2t$.

10. W is a subspace of $\mathbb{R}^4$ by Theorem 2, because W is the set of solutions of the homogeneous system

$$\begin{aligned} a + 3b - c \phantom{{}-d} &= 0 \\ a + b + c - d &= 0 \end{aligned}$$

12. If $(b - 5d, 2b, 2d + 1, d)$ were the zero vector, then $2d + 1 = 0$ and $d = 0$, which is impossible. So **0** is not in W, and W is not a subspace.

14. $W = \text{Col } A$ for $A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \\ 3 & -6 \end{bmatrix}$, so W is a vector space by Theorem 3.

16. $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}$

18. a. 3 **b.** 4 **20. a.** 5 **b.** 1

22. $\begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}$ in Nul A, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in Col A. Other answers are possible.

24. **w** is in both Nul A and Col A. $A\mathbf{w} = \mathbf{0}$, and $\mathbf{w} = -\frac{1}{2}\mathbf{a}_1 + \mathbf{a}_2$.

25. a. True, by the definition before Example 1.
b. False. See Theorem 2.
c. True. See the remark just before Example 4.
d. False. The equation $A\mathbf{x} = \mathbf{b}$ must be consistent *for every* **b**. See #7 in the table on p. 232.
e. True. See Fig. 2. (A subspace is itself a vector space.)
f. True. See the remark after Theorem 3.

26. a. True. See Theorem 2. (A subspace is itself a vector space.)
b. True. See Theorem 3.
c. False. See the box after Theorem 3.
d. True. See the paragraph after the definition of a linear transformation.
e. True. See Fig. 2. (A subspace is itself a vector space.)
f. True. See the paragraph before Example 8.

28. The two systems have the form $A\mathbf{x} = \mathbf{v}$ and $A\mathbf{x} = 5\mathbf{v}$. Since the first system is consistent, **v** is in Col A. Since Col A is a subspace of $\mathbb{R}^3$, $5\mathbf{v}$ is also in Col A. Thus the second system is consistent.

30. The zero vector $\mathbf{0}_W$ of W is in the range of T, because the linear transformation maps the zero vector of V to $\mathbf{0}_W$. Typical vectors in the range of T are $T(\mathbf{x})$ and $T(\mathbf{w})$, where **x**, **w** are in V. Since T is a linear transformation,

$T(\mathbf{x}) + T(\mathbf{w}) = T(\mathbf{x} + \mathbf{w})$ In the range of T

Thus the range of T is closed under vector addition. Also, for any scalar c, $c \cdot T(\mathbf{x}) = T(c\mathbf{x})$, since T is a linear transformation. Thus $c \cdot T(\mathbf{x})$ is in the range of T, so the range is closed under scalar multiplication. Hence the range of T is a subspace of W.

32. $\mathbf{p}_1(t) = t$, $\mathbf{p}_2(t) = t^2$. The range of T is $\left\{\begin{bmatrix} a \\ a \end{bmatrix} : a \text{ real}\right\}$.

34. The kernel of T is $\{\mathbf{0}\}$.

36. Since Z is a subspace of W, the zero vector $\mathbf{0}_W$ of W is in Z. Because T is linear, T maps the zero vector $\mathbf{0}_V$ of V to $\mathbf{0}_W$. Thus $\mathbf{0}_V$ is in $U = \{\mathbf{x} : T(\mathbf{x}) \text{ is in } Z\}$. Now take $\mathbf{u}_1$, $\mathbf{u}_2$ in U. Since T is linear,

$$T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2) \qquad (*)$$

By definition of Z, $T(\mathbf{u}_1)$ and $T(\mathbf{u}_2)$ are in Z, and so the sum on the right of $(*)$ is in Z because Z is a subspace. This proves that $\mathbf{u}_1 + \mathbf{u}_2$ is in U, so U is closed under vector addition. For any scalar c, $c \cdot T(\mathbf{u}_1)$ is in Z because Z is a subspace. Since T is linear, $T(c\mathbf{u}_1)$ is in Z. Hence $c\mathbf{u}_1$ is in U. Thus U is a subspace of V.

37. [M] **w** is in Col A. In fact, $\mathbf{w} = A\mathbf{x}$ for

$\mathbf{x} = (1/95, -20/19, -172/95, 0)$

w is not in Nul A because $A\mathbf{w} = (14, 0, 0, 0)$.

38. [M] **w** is in Col A and in Nul A because $\mathbf{w} = A\mathbf{x}$ for $\mathbf{x} = (-2, 3, 0, 1)$, and $A\mathbf{w} = (0, 0, 0, 0)$.

39. [M] The reduced echelon form of A is

$$\begin{bmatrix} 1 & 0 & 1/3 & 0 & 10/3 \\ 0 & 1 & 1/3 & 0 & -26/3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a. Most students will row reduce $[\,B \quad \mathbf{a}_3\,]$ and $[\,B \quad \mathbf{a}_5\,]$ to show that the equations $B\mathbf{x} = \mathbf{a}_3$ and $B\mathbf{x} = \mathbf{a}_5$ are consistent. You can use a discussion of this part to lead into Examples 8 and 9 in Section 4.3.

b. The method of Example 3 produces $(-1/3, -1/3, 1, 0, 0)$ and $(-10/3, 26/3, 0, 4, 1)$.

c. This part reviews Section 1.9. An echelon form of A shows that the columns of A are linearly dependent and do not span $\mathbb{R}^4$. By Theorem 12 in Section 1.9, T is not one-to-one and T does not map $\mathbf{R}^5$ onto $\mathbf{R}^4$.

40. **[M]** Row reduction of $[\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad -\mathbf{v}_3 \quad -\mathbf{v}_4 \quad \mathbf{0}\,]$ yields

$$\begin{bmatrix} 1 & 0 & 0 & -10/3 & 0 \\ 0 & 1 & 0 & 26/3 & 0 \\ 0 & 0 & 1 & -4 & 0 \end{bmatrix}$$

The general solution is a multiple of $(10, -26, 12, 3)$. One choice for $\mathbf{w}$ is $10\mathbf{v}_1 - 26\mathbf{v}_2$ $(= 12\mathbf{v}_3 + 3\mathbf{v}_4)$, which is $(24, -48, -24)$. Another choice is $\mathbf{w} = (1, -2, -1)$.

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2. No, the set is linearly dependent because the zero vector is in the set. The columns of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ do not span $\mathbb{R}^3$, by the Invertible Matrix Theorem.

4. Yes. See Example 5 for an example of a justification.

6. No, $\begin{bmatrix} 1 & -4 \\ 2 & -5 \\ -3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$. The matrix does not have a pivot in each row, so its columns do not span $\mathbb{R}^3$ and hence do not form a basis. However, the columns are linearly independent because they are not multiples. (More precisely, neither column is a multiple of the other.)

8. No, the vectors are linearly dependent because there are more vectors than entries in each vector. However, the vectors do span $\mathbb{R}^3$.

10. $\begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ **12.** $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

14. Basis for Nul A: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix}$

Basis for Col A: $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix}$

16. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ **18.** **[M]** $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$

20. The three simplest answers are $\{\mathbf{v}_1, \mathbf{v}_2\}$ or $\{\mathbf{v}_1, \mathbf{v}_3\}$ or $\{\mathbf{v}_2, \mathbf{v}_3\}$. Other answers are possible.

21. **a.** False. The zero vector by itself is linearly dependent. See the paragraph preceding Theorem 4.
b. False. The set $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ must also be linearly independent. See the definition of a basis.
c. True. See Example 3.
d. False. See the subsection "Two Views of a Basis."
e. False. See the box before Example 9.

22. **a.** False. The subspace spanned by the set must also coincide with H. See the definition of a basis.
b. True, by the Spanning Set Theorem, applied to V instead of H. (V is nonzero because the spanning set uses nonzero vectors.)
c. True. See the subsection "Two Views of a Basis."
d. False. See two paragraphs before Example 8.
e. False. See the warning after Theorem 6.

24. Let $A = [\,\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_n\,]$. Since A is square and its columns are linearly independent, its columns also span $\mathbb{R}^n$, by the Invertible Matrix Theorem. So $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$.

26. A basis is $\{\sin t, \sin 2t\}$ because this set is linearly independent (by inspection), and $\sin t \cos t = \frac{1}{2}\sin 2t$, as pointed out in Example 2.

28. $\{e^{-bt}, te^{-bt}\}$. The set is linearly independent because neither function is a *scalar* multiple of the other, and the set spans H.

30. There are more vectors than there are entries in each vector. By Theorem 8 in Section 1.6, the set is linearly dependent and therefore cannot be a basis for $\mathbb{R}^n$.

32. Suppose that $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ is linearly dependent. Then there exist $c_1, \ldots, c_p$, not all zero, such that

$$c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p) = \mathbf{0}$$

Since T is linear and $\mathbf{0} = T(\mathbf{0})$,

$$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = T(\mathbf{0})$$

By hypothesis, T is one-to-one, so this equation implies that $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$, which shows that $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent.

34. By inspection, $\mathbf{p}_3 = \mathbf{p}_1 + \mathbf{p}_2$, or $\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 = \mathbf{0}$. By the Spanning Set Theorem, Span$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ = Span$\{\mathbf{p}_1, \mathbf{p}_2\}$. Since neither $\mathbf{p}_1$ nor $\mathbf{p}_2$ is a multiple of the other, they are linearly independent and hence $\{\mathbf{p}_1, \mathbf{p}_2\}$ is a basis for Span$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.

36. **[M]** Row reducing $[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]$ shows that $\mathbf{u}_1$ and $\mathbf{u}_2$ are the pivot columns of this matrix. Thus $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for H.

Row reducing $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ shows that $\mathbf{v}_1$ and $\mathbf{v}_2$ are the pivot columns of this matrix. Thus $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for K.

Row reducing $[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ shows that $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{v}_1$ are the pivot columns of this matrix. Thus $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1\}$ is a basis for $H + K$.

38. **[M]** For example, writing

$$c_1 \cdot 1 + c_2 \cdot \cos t + c_3 \cdot \cos^2 t + c_4 \cdot \cos^3 t + c_5 \cdot \cos^4 t + c_6 \cdot \cos^5 t + c_7 \cdot \cos^6 t = 0$$

with $t = 0, .1, .2, .3, .4, .5, .6$ gives a 7×7 coefficient matrix A for the homogeneous system $A\mathbf{c} = \mathbf{0}$. The matrix A is invertible, so the system $A\mathbf{c} = \mathbf{0}$ has only the trivial solution and $\{1, \cos t, \cos^2 t, \cos^3 t, \cos^4 t, \cos^5 t, \cos^6 t\}$ is a linearly independent set of functions.

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2. $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ **4.** $\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$ **6.** $\begin{bmatrix} -6 \\ 2 \end{bmatrix}$ **8.** $\begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$

10. $\begin{bmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{bmatrix}$ **12.** $\begin{bmatrix} -7 \\ 5 \end{bmatrix}$ **14.** $\begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}$

15. **a.** True, by definition of the $\mathcal{B}$-coordinate vector.
b. False. See equation (4).
c. False. $\mathbb{P}_3$ is isomorphic to $\mathbb{R}^4$. See Example 5.

16. **a.** True. See Example 2.
b. False. By definition, the coordinate mapping goes in the reverse direction.
c. True, when the plane passes through the origin, as in Example 7.

18. Since $\mathbf{b}_1 = 1 \cdot \mathbf{b}_1 + 0 \cdot \mathbf{b}_2 + \cdots + 0 \cdot \mathbf{b}_n$, the $\mathcal{B}$-coordinate vector of $\mathbf{b}_1$ is

$$[\mathbf{b}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{e}_1$$

For each k, $\mathbf{b}_k = 0 \cdot \mathbf{b}_1 + \cdots + 1 \cdot \mathbf{b}_k + \cdots + 0 \cdot \mathbf{b}_n$, so $[\mathbf{b}_k]_{\mathcal{B}} = (0, \ldots, 1, \ldots, 0) = \mathbf{e}_k$.

20. For $\mathbf{w}$ in V, there exist scalars $k_1, \ldots, k_4$ such that

$$\mathbf{w} = k_1\mathbf{v}_1 + \cdots + k_4\mathbf{v}_4 \quad (1)$$

because $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ spans V. Also, because the set is linearly dependent, there exist scalars $c_1, \ldots, c_4$, not all zero, such that

$$\mathbf{0} = c_1\mathbf{v}_1 + \cdots + c_4\mathbf{v}_4$$

Adding gives

$$\mathbf{w} = \mathbf{w} + \mathbf{0} = (k_1 + c_1)\mathbf{v}_1 + \cdots + (k_4 + c_4)\mathbf{v}_4$$

At least one of the weights here differs from the corresponding weight in (1) because at least one of the c_i is nonzero. So $\mathbf{w}$ is expressed in more than one way as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_4$.

22. Let $P_{\mathcal{B}} = [\,\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n\,]$. Then $P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$ and $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$. As mentioned in the text, the correspondence $\mathbf{x} \mapsto P_{\mathcal{B}}^{-1}\mathbf{x}$ is the coordinate mapping, so the desired matrix is $A = P_{\mathcal{B}}^{-1}$.

24. Given $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, let $\mathbf{u} = y_1\mathbf{b}_1 + \cdots + y_n\mathbf{b}_n$. Then, by definition, $[\mathbf{u}]_{\mathcal{B}} = \mathbf{y}$. So the coordinate mapping transforms $\mathbf{u}$ into $\mathbf{y}$. Since $\mathbf{y}$ was arbitrary, the coordinate mapping is onto.

26. $\mathbf{w}$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_p$ if and only if there exist scalars $c_1, \ldots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p \quad (2)$$

Since the coordinate mapping is linear,

$$[\mathbf{w}]_{\mathcal{B}} = c_1[\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_p[\mathbf{u}_p]_{\mathcal{B}} \quad (3)$$

Conversely, (2) implies (3) because the coordinate mapping is one-to-one. Thus $\mathbf{w}$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_p$ if and only if (3) holds for some $c_1, \ldots, c_p$, which is equivalent to saying that $[\mathbf{w}]_{\mathcal{B}}$ is a linear combination of $[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_p]_{\mathcal{B}}$.

Note: Students need to be urged to *write*, not just to compute, in Exercises 27–34. The language in the *Study Guide* solution of Exercise 31 provides a model for the students. In Exercise 32, students may have difficulty distinguishing between the two isomorphic vector spaces, sometimes giving a vector in $\mathbb{R}^3$ as the answer for part (b).

28. Linearly dependent because the coordinate vectors

$$\begin{bmatrix} 1 \\ 0 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \\ 0 \end{bmatrix} \text{ are linearly dependent.}$$

30. Linearly dependent. The coordinate vectors

$$\begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -12 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ -4 \end{bmatrix} \text{ are linearly dependent.}$$

32. a. The coordinate vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$ span $\mathbb{R}^3$.

Thus these three vectors form a basis for $\mathbb{R}^3$ by the Invertible Matrix Theorem. Because of the isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$, the corresponding polynomials form a basis for $\mathbb{P}_2$.

b. Since $[\mathbf{q}]_{\mathcal{B}} = (-3, 1, 2)$, one may compute

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -8 \end{bmatrix}$$

and $\mathbf{q} = 1 + 3t - 8t^2$.

34. [M] The coordinate vectors $\begin{bmatrix} 5 \\ -3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 \\ 1 \\ 8 \\ -6 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 5 \\ 0 \end{bmatrix},$

$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are linearly dependent. Because of the isomorphism between $\mathbb{R}^4$ and $\mathbb{P}_3$, the corresponding polynomials are linearly dependent and therefore cannot form a basis for $\mathbb{P}_3$.

36. [M] Row reduction of $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ shows that there is a pivot in each column, so the columns are linearly independent and hence form a basis for the subspace H which they span.

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

38. [M] $\begin{bmatrix} 1.30 \\ .75 \\ 1.60 \end{bmatrix}$