Homework Set 9

(sect 5.1 – 5.4)

1. Is $\lambda = 3$ an eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$? If so, find one corresponding eigenvector.

2. Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$? If so, find the corresponding eigenvalue.

3. Find a basis for the eigenspace of $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix}$ corresponding to $\lambda = -2$.

4. For $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, find one eigenvalue with no calculation. Justify your answer.

For questions 5 through 7, find the characteristic polynomial and the eigenvalues of each matrix.

5. $A = \begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

7. $A = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}$

For questions 8 and 9, list the eigenvalues of the matrices, repeated according to their multiplicities.

8. $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix}$

9. $A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 & 0 \\ 3 & 8 & 0 & 0 & 0 \\ 0 & -7 & 2 & 1 & 0 \\ -4 & 1 & 9 & -2 & 3 \end{bmatrix}$

10. Let $A = PDP^{-1}$ and compute A^4 where $P = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$.

11. Use the factorization $A = PDP^{-1}$ to compute A^k, where k represents some positive integer. $A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$

12. The matrix A is factored in the form $A = PDP^{-1}$. Use the Diagonalization Theorem in section 5.3 to find the eigenvalues of A and a basis for each eigenspace.

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -2 \end{bmatrix}$$

For questions 13 through 15, diagonalize the given matrices if possible.

13. $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

14. $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$, where $\lambda = 2,1$

16. Let $\mathcal{E} = \{\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}\}$ be the standard basis for $\mathbb{R}^3$, $\mathcal{B} = \{\boldsymbol{b_1}, \boldsymbol{b_2}, \boldsymbol{b_3}\}$ be a basis for vector space V, and $T: \mathbb{R}^3 \to V$ be a linear transformation with the property that

$$T(x_1, x_2, x_3) = (x_3 - x_2)\boldsymbol{b_1} - (x_1 + x_3)\boldsymbol{b_2} + (x_1 - x_2)\boldsymbol{b_3}$$

a. Compute $T(\boldsymbol{e_1})$, $T(\boldsymbol{e_2})$, and $T(\boldsymbol{e_3})$.

b. Compute $[T(\boldsymbol{e_1})]_{\mathcal{B}}$, $[T(\boldsymbol{e_2})]_{\mathcal{B}}$, and $[T(\boldsymbol{e_3})]_{\mathcal{B}}$.

c. Find the matrix for T relative to $\mathcal{E}$ and $\mathcal{B}$.

17. Let $\mathcal{A} = \{\boldsymbol{a_1}, \boldsymbol{a_2}\}$ and $\mathcal{B} = \{\boldsymbol{b_1}, \boldsymbol{b_2}\}$ be bases for vector spaces V and W, respectively. Let $T: V \to W$ be a linear transformation with the property that

$$T(\boldsymbol{a_1}) = 2\boldsymbol{b_1} - 3\boldsymbol{b_2}, \qquad T(\boldsymbol{a_2}) = -4\boldsymbol{b_1} + 5\boldsymbol{b_2}$$

Find the matrix for T relative to $\mathcal{A}$ and $\mathcal{B}$.

18. Let $T: \mathbb{P}_2 \rightarrow \mathbb{P}_4$ be the transformation that maps a polynomial $\boldsymbol{p}(t)$ into the polynomial $\boldsymbol{p}(t) + t^2\boldsymbol{p}(t)$.
    a. Find the image of $\boldsymbol{p}(t) = 2 - t + t^2$.

    b. Show that T is a linear transformation.

    c. Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$.

19. Define $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(\boldsymbol{x}) = A\boldsymbol{x}$. Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal, where $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$.

20. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ and $\mathcal{B} = \{\boldsymbol{b_1}, \boldsymbol{b_2}\}$ for $\boldsymbol{b_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{b_2} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$. Define $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(\boldsymbol{x}) = A\boldsymbol{x}$.

    a. Verify that $\boldsymbol{b_1}$ is an eigenvector of A, but A is not diagonalizable.

    b. Find the $\mathcal{B}$-matrix for T.