Practice Problems for Final

chapters :	1.1 – 1.5, 1.7 – 1.9	(see test 1)
chapters :	2.1 – 2.5, 3.1 – 3.2	(see midterm)
chapters :	3.3, 4.1 – 4.5	(see test 3)
chapters :	4.6 – 4.7, 5.1 – 5.4	(see below)

1. $$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 9 & 2 \\ 0 & 1 & -1 & 0 & 7 & 3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Find rank A and Nulity A

$= 3$ $\quad = 3$

b) Find dim row A and dim Col A^T

$= 3$ $\quad = 3$

c) What is Col A? Row A? Nul A?

$\text{Col } A = \text{span}\{a_1, a_2, a_3, a_4, a_5, a_6\}$

$\text{row } A = \text{span}\{r_1, r_2, r_3, r_4, r_5\}$

$$\text{Nul } A = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

d) find a basis for each: Col A, row A, Nul A

basis for Col A $= \{a_1, a_2, a_4\}$

$$\text{basis for Row } A = \left\{ \begin{array}{l} (1, 1, -3, 7, 9, -9) \\ (0, 1, -1, 3, 4, -3) \\ (0, 0, 0, 2, -2, -4) \end{array} \right\}$$

$$\text{basis for Nul } A = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

2\. a) If the null space of a 7×6 matrix A is 5-dimensional, what is the dimension of the column space of A?

$$\dim \operatorname{Col} A = 1$$

b) If A is a 6×8 matrix what is the <u>smallest</u> posible dimension of $\operatorname{Nul} A$?

$$\text{Nulity } A = 8 - \text{rank } A \quad , \quad 0 \leq \text{rank } A \leq 6$$

$$\text{Nulity } A = 8 - 6 = 2$$

c) If A is a 5×6 matrix with 4 pivot columns, what is rank A? What is Nulity A? Is $\operatorname{Col} A = \mathbb{R}^4$? Why or Why not?

$$\text{rank } A = 4 \quad , \quad \text{Nulity } A = 6 - 4 = 2$$

$\operatorname{Col} A \neq \mathbb{R}^4$ b/c each column in A has 5 elements so $\operatorname{Col} A \subseteq \mathbb{R}^5$

3\. Let $\mathcal{B} = \{b_1, b_2, b_3\}$ and $\mathcal{C} = \{c_1, c_2, c_3\}$ be bases for a vector space V, and we know that

$$c_1 = 2b_1 - b_2 + b_3 \quad , \quad c_2 = 3b_2 + b_3 \quad , \quad c_3 = -3b_1 + 2b_3$$

a) Find the change of coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$

$$\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

b) Find $[x]_{\mathcal{B}}$ for $x = c_1 - 2c_2 + 2c_3$

$$[x]_{\mathcal{B}} = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}$$

Since

$$x = (2b_1 - b_2 + b_3) - 2(3b_2 + b_3) + 2(-3b_1 + 2b_3)$$

$$= -4b_1 - 7b_2 + 3b_3$$

4. $\mathcal{B}$ and $\mathcal{C}$ are bases of $\mathbb{R}^2$. Find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and from $\mathcal{C}$ to $\mathcal{B}$.

$\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ where

$$b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} c_1 & c_2 & b_1 & b_2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 3/5 & -1/5 \\ 0 & 1 & -1/5 & 2/5 \end{array}\right] \Rightarrow \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}^{-1}$$

$$\left[\begin{array}{cc|cc} b_1 & b_2 & c_1 & c_2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right] \Rightarrow \underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix}$$

5. $$A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$$

a) find the characteristic equation of A

$$\det\begin{pmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{pmatrix} = (7-\lambda)(-1-\lambda) - 9 = \lambda^2 - 6\lambda - 16 = 0$$

OR $(\lambda - 8)(\lambda + 2) = 0$

b) give the eigenvalues of A

$$\lambda = 8, -2$$

c) find the eigenspace associated to each λ

$\underline{\lambda = 8}$ $\quad \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 8a \\ 8b \end{bmatrix} \Rightarrow \left.\begin{matrix} 7a + 3b = 8a \\ 3a - b = 8b \end{matrix}\right\} \Rightarrow a = 3b$

$$E_{\lambda=8} = \text{span}\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$$

$\underline{\lambda = -2}$ $\quad \begin{bmatrix} 7-(-2) & 3 \\ 3 & -1-(-2) \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} 9a + 3b = 0 \\ 3a + b = 0 \end{matrix}\right\} \Rightarrow b = -3a$

$$E_{\lambda=-2} = \text{span}\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\}$$

d) ~~give~~ what is the dimension of each eigenspace?

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6. $A = \begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}$

a) find the characteristic equation of A

$$\begin{vmatrix} -7-\lambda & -16 & 4 \\ 6 & 13-\lambda & -2 \\ 12 & 16 & 1-\lambda \end{vmatrix} = -75 + 5\lambda + 7\lambda^2 - \lambda^3 = 0$$

OR $(5-\lambda)^2(-3-\lambda) = 0$

b) what are the eigenvalues of A?

$\lambda = 5, -3$

c) find the eigenspace(s) associated to the eigenvalues of A.

$\lambda = 5$ $\begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5a \\ 5b \\ 5c \end{bmatrix} \Rightarrow 3a+4b=c \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} a + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} b$

$\lambda = -3$ $\begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -3a \\ -3b \\ -3c \end{bmatrix} \Rightarrow \begin{matrix} a+c=0 \\ 2b=c \end{matrix} \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} b$

d) What are the dimension(s) of the eigenspace(s)?

$E_{\lambda=5}$ has dim 2

$E_{\lambda=-3}$ has dim 1

e) Is A diagonalizable? Why?

yes, b/c the sum of the dim of the eigenspaces is 3 (A is 3×3)

f) If A is diagonalizable, give it's diagonal factorization. If A is not diagonalizable, give/list a eigenvector coresponding to each eigenvalue.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix}^{-1}$$

7. $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

If A is diagonalizable, find P and D such that $A = PDP^{-1}$.
If not diagonalizable, explain why.

① $\lambda = 4, 2$

② $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \\ 2c \\ 2d \end{bmatrix} \Rightarrow \begin{matrix} 4a = 2a \\ 4b = 2b \\ 2c = 2c \\ a + 2d = 2d \end{matrix} \Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} c + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} d$

$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 4a \\ 4b \\ 4c \\ 4d \end{bmatrix} \Rightarrow \begin{matrix} 4a = 4a \\ 4b = 4b \\ 2c = 4c \\ a + 2d = 4d \end{matrix} \Rightarrow \begin{matrix} c = 0 \\ a = 2d \end{matrix}$

$\Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2d \\ b \\ 0 \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} b + \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} d$

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$$A = \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}}_{\text{"}P} \underbrace{\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_{\text{"}D} \underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \end{bmatrix}}_{\text{"}P^{-1}}$$

8. $A = PDP^{-1}$ where

$$A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

(a) compute A^{10} (Note: answer needs to be a single matrix)

$$A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2^{10} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2^{10} & -2^{10} \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} (3\cdot 2^{10}-2) & (-3\cdot 2^{10}+3) \\ (2^{11}-2) & (-2^{11}+3) \end{bmatrix} = \begin{bmatrix} 3070 & -3069 \\ 2046 & -2045 \end{bmatrix}$$

(b) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation where $T(\bar{x}) = A\bar{x}$. Is there a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is diagonal? If so, what is $\mathcal{B}$ and what is $[T]_{\mathcal{B}}$?

yes.

$$[T]_{\mathcal{B}} = D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

where

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

9. Compute A^5 where $A = \begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix}$

Step 1: diagonalize A if possible

$$A = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 3/4 & 1/4 \\ 1/4 & -1/4 \end{bmatrix}}_{P^{-1}} \qquad \lambda = 6, 2$$

Step 2:

$$A^5 = PD^5P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 7776 & 0 \\ 0 & 32 \end{bmatrix} \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & -1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 5840 & 1936 \\ 5808 & 1968 \end{bmatrix}$$

10. Let $\mathcal{B} = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ be bases for the vector spaces V and W. Let $T: V \to W$ be a Linear transformation where

$$T(b_1) = 2c_1 - 4c_2 \quad \& \quad T(b_2) = 3c_1 + 7c_2$$

find the matrix for T relative to $\mathcal{B}$ and C.

$$M = \left[\, [T(b_1)]_C \quad [T(b_2)]_C \,\right]$$

$$= \begin{bmatrix} 2 & 3 \\ -4 & 7 \end{bmatrix}$$

11. find the $\mathcal{B}$-matrix for the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ where $T(x) = Ax$ when

$$A = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathcal{B} = \{b_1, b_2\}$$

$$T(b_1) = Ab_1 = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$T(b_2) = Ab_2 \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$[T(b_1)]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

where $c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$$[T(b_2)]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Where $c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

ie: $\left[\begin{array}{cc|c} 3 & -1 & 5 \\ 2 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$

so

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$