1.2: Row Reduction & Echelon Forms

nonzero row/column – at least 1 nonzero entry in the row/column

leading entry – the 1^{st} one

Echelon Form (Row Echelon Form)

1. all nonzero rows are above all zero rows
2. each leading entry of a row is in a column to the right of the leading entry in the row above it
3. all entries in a column below a leading entry are zero

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form (RREF)

4. the leading entry in each nonzero row is 1
5. Each leading 1 is the only nonzero entry in its column

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Def: Echelon Matrix – a matrix in echelon form

Note: any matrix may be row reduced

Thrm 1: Uniqueness of the REF
Each matrix is row equivalent to exactly 1 reduced echelon matrix

Pivot Positions

Def: a location in matrix A corresponding to a leading 1 in the reduced echelon form of A. These positions don't change

Def: Pivot Column
the column of A which contains a pivot

$$\begin{bmatrix} -2 & -3 & 0 & 3 & -1 \\ -1 & -2 & -1 & 3 & 1 \\ 1 & 4 & 5 & -9 & -7 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

find the pivots & pivot columns

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

want 1 to be in top left ~~cor~~ corner

$$\begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

zeroed out the 1^{st} Column

$$\begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow R_3 - \frac{5}{2}R_2 \\ \leftarrow R_4 + \frac{3}{2}R_2 \end{array}$$

$$\begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & \boxed{-5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivots

Original:

$$\begin{bmatrix} \boxed{-2} & -3 & 0 & 3 & -1 \\ -1 & \boxed{-2} & -1 & 3 & 1 \\ 1 & 4 & \boxed{5} & \boxed{-9} & -7 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\begin{array}{ccccc} \uparrow & \uparrow & & \uparrow & \end{array}$$

pivot columns

Reduced Row Echelon (Algorithm)

$$\begin{bmatrix} 0 & 4 & 1 & -5 & 0 \\ -3 & -7 & 0 & 1 & 3 \\ 3 & 6 & -9 & 12 & 15 \end{bmatrix}$$

Begin w/ left column

$$\begin{bmatrix} 3 & 6 & -9 & 12 & 15 \\ -3 & -7 & 0 & 1 & 3 \\ 0 & 4 & 1 & -5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & -9 & 12 & 15 \\ 0 & -1\cancel{3} & -9 & 13 & 18 \\ 0 & 4 & 1 & -5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & 4 & 5 \\ 0 & -1 & -9 & 13 & 18 \\ 0 & 0 & -35 & 47 & 72 \end{bmatrix}$$

want 1 in top left corner

$$\begin{bmatrix} \boxed{1} & 2 & -3 & 4 & 5 \\ 0 & \boxed{1} & 9 & -13 & -18 \\ 0 & 0 & \boxed{1} & -47/35 & -72/35 \end{bmatrix}$$

want 1 in pivot possitions

Want pivot columns to be all zeros (except pivots)

$$\begin{bmatrix} 1 & 0 & -31 & 30 & 41 \\ 0 & 1 & 9 & -13 & -18 \\ 0 & 0 & 1 & -47/35 & -72/35 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1457/35 & -2232/35 \\ 0 & 1 & 0 & 423/35 & 648/35 \\ 0 & 0 & 1 & -47/35 & -72/35 \end{bmatrix}$$

this is the rref of the matrix

Solutions of Linear Systems

reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

interpretation:

$$\begin{aligned} x_1 + \quad\quad + x_3 &= 2 \\ x_2 - 4x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\Longrightarrow \begin{cases} x_1 = 2 - x_3 \\ x_2 = 4x_3 \\ x_3 = \text{free variable} \end{cases}$$

general Solution (infinitely many)

Free Variable:

a variable where you can pick its value.

Parametric Form:

a solution in parametric form means to write out $x_1 =$

$\vdots$

$x_n =$

where $x_i = \#$ or an expression

ex: The augment matrix of a Linear system is shown below. Find its solution.

$$\begin{bmatrix} 1 & 2 & -4 & 1 & -1 & 3 \\ 0 & 0 & 5 & -2 & 15 & 0 \\ 0 & 0 & 0 & 0 & 3 & 5 \end{bmatrix}$$

Step 1: put into reduced echelon form

$$\begin{bmatrix} 1 & 2 & -4 & 1 & -1 & 3 \\ 0 & 0 & 1 & -2/5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5/3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -4 & 1 & 0 & \frac{14}{3} \\ 0 & 0 & 1 & -2/5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & 5/3 \end{bmatrix} \quad \begin{array}{l} R_1 = R_1 + R_3 \\ R_2 = R_2 - 3R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3/5 & 0 & -46/3 \\ 0 & 0 & 1 & -2/5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & 5/3 \end{bmatrix} \quad R_1 = R_1 + 4R_3$$

Step 2: intepret

$$\begin{cases} x_1 + 2x_2 - \frac{3}{5}x_4 = -46/3 \\ x_3 - 2/5\, x_4 = -5 \\ x_5 = 5/3 \end{cases}$$

Step 3: Solve

$x_1 = -2x_2 + 3/5\, x_4 - 46/3$

$x_2 =$ free

$x_3 = 2/5\, x_4 - 5$

$x_4 =$ free

$x_5 = 5/3$

Existence & Uniqueness

unique $\longrightarrow$ no free variables

Existance $\rightarrow$ consistancy (ie $0 \neq 1$)

Step 1:

Write the system's Augmented matrix

Step 2:

~~row~~ put the matrix into row echelon form (doesn't need to be in reduced row echelon form)

Step 3: Existence

check for rows : $[0\ 0 \ldots 0\ b]$

$\Rightarrow$ not consistant

or doesn't exist

Step 4: Uniqueness

check for fee variables

Examples (reduced augmented matrices):

$$\begin{bmatrix} 3 & 4 & 5 & 1 & 7 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 4 & -5 & 1 \\ 0 & 0 & 1 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 7 & -8 & 1 & 2 \\ 0 & 2 & -9 & 13 & 4 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 4 & 0 & -2 & 7 & 4 & 1 \\ 0 & 5 & 1 & 8 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Thrm 2: Existance + Uniqueness

a Linear system is unique iff the rightmost column is not a pivot column
(ie: in RREF we don't have $[0\ 0 \ldots 0\ b]$ & $b \neq 0$)

If the system is consistant then
(i) unique solution - no free variables
(ii) infinitely many solutions - free var.

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Using Row Reduction to Solve a Linear System

1. get the augment matrix

2. row reduce to get the echelon form
if inconsistant → stop
if consistant → go to step 3

3. get the reduced row echelon form

4. interpret : get the equations

5. intepret : solve for $x_i = \square$
put into parametric form