1.3: vector Equations

Vectors in $\mathbb{R}^2$

column vectors:

$$\bar{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \bar{v} = \begin{bmatrix} -4 \\ 0 \end{bmatrix} \qquad \bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

row vectors:

$$\bar{a} = \begin{bmatrix} 0 , 1 \end{bmatrix} , \quad \bar{b} = \begin{bmatrix} 1/4 & -3 \end{bmatrix}$$

equal?

$$\begin{bmatrix} 41 \\ -5 \end{bmatrix} \neq \begin{bmatrix} -5 \\ 41 \end{bmatrix}$$

sum?

$$\begin{bmatrix} -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} -1+7 \\ 3+2 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

Scaler multiple:

$$2\begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 2\cdot 5 \\ 2\cdot -3 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \end{bmatrix}$$

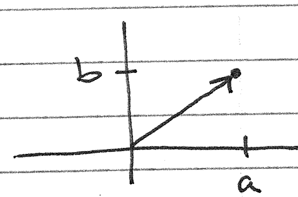
Example:

$$u = \begin{bmatrix} 1 \\ -2 \end{bmatrix} , \quad v = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

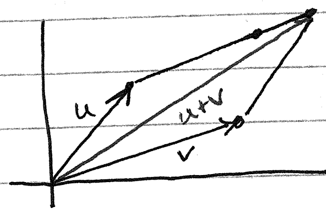
$$5u - 8v = 5\begin{bmatrix} 1 \\ -2 \end{bmatrix} - 8\begin{bmatrix} 3 \\ 11 \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix} - \begin{bmatrix} 24 \\ 88 \end{bmatrix} = \begin{bmatrix} -19 \\ -98 \end{bmatrix}$$

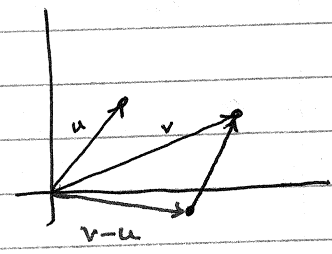
Geometric Description

$\begin{bmatrix} a \\ b \end{bmatrix}$ a ray from (0,0) to (a,b)



Parallelogram Rule for addition





Vectors in $\mathbb{R}^n$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Zero vector: $\bar{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

Properties of $\mathbb{R}^n$:

vectors: $u, v, w \in \mathbb{R}^n$ and scalers c & d

1) $u+v = v+u$
2) $(u+v)+w = u+(v+w)$
3) $u+0 = 0+u = u$
4) $u+(-u) = -u+u = 0$
5) $c(u+v) = cu+cv$
6) $(c+d)u = cu+du$
7) $c(du) = (cd)u$
8) $1 \cdot u = u$

Linear Combinations

$$y = c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots + c_n v_n$$

c_i - scalers "weights"

v_i - vectors

example:

$$a_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

determine if $\exists$ scalers c_1 & c_2 $\ni$

$$c_1 a_1 + c_2 a_2 = b$$

Solution:

$$\begin{cases} 3c_1 + 2c_2 = 1 \\ -c_1 + 5c_2 = 0 \end{cases}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ -1 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 5 & 0 \\ 0 & 17 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 1/17 \end{bmatrix} \Longrightarrow \begin{matrix} x_1 = 5x_2 \\ x_2 = 1/17 \end{matrix}$$

So $c_1 = 5/17$, $c_2 = 1/17$

Vector Equation

$$x_1 \bar{a}_1 + x_2 \bar{a}_2 + \dots + x_n \bar{a}_n = \bar{b}$$

has the same solution as the augmented matrix $[a_1\ a_2 \dots a_n\ b]$

ie: $\bar{b}$ can be formed by a Linear combo of the $\bar{a}_i$

$\iff$ $\exists$ a solution to the system

Span$\{v_1, \dots, v_p\}$

Span$\{v_1, \dots, v_p\}$ = set/collection of all linear combinations of the vectors v_i which are all in $\mathbb{R}^n$

So for all vectors $y \in$ Span$\{v_i\}$ we have

$$y = c_1 v_1 + \cdots + c_p v_p$$

<u>ex</u>:

$$a_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}, \quad b = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

is $b \in$ Span$\{a_1, a_2\}$?

does $c_1 a_1 + c_2 a_2 = b$ have a solution?

$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

no solution

Ans : $b \notin$ Span$\{a_1, a_2\}$

1.4: The Matrix Equation $Ax=b$

The Matrix Equation

$$A\bar{x} = [a_1\ a_2\ \dots\ a_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\bar{a}_1 + \dots + x_n\bar{a}_n$$

A - matrix $(m \times n)$
$\bar{x}$ - vector $(n \times 1)$
a_i - columns of A & coef of x_i

& $A\bar{x}=\bar{b}$ has the same solution set as the augmented matrix: $[a_1\ a_2 \dots a_n\ b]$

$A\bar{x}=\bar{b}$ has a solution $\Leftrightarrow$ $b \in \text{span}\{a_1, \dots, a_n\}$

Thrm: TFAE
1) $\forall b \in \mathbb{R}^m$, $A\bar{x}=\bar{b}$ has a solution
2) $\forall b \in \mathbb{R}^m$, $b \in \text{Span}\{a_1, \dots, a_m\}$
3) $\text{span}\{a_1, \dots, a_n\} = \mathbb{R}^m$
4) A has a pivot in every row

Existance of Solutions

when is $Ax=b$ consistant $\forall b$?

ex: $A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 1 & -5 \\ -3 & 2 & -4 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 3 & 0 & b_1 \\ -2 & 1 & -2 & b_2 \\ -3 & 5 & -4 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & b_1 \\ 0 & 7 & -2 & (2b_1+b_2) \\ 0 & 14 & -4 & (3b_1+b_3) \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & b_1 \\ 0 & 7 & -2 & (2b_1+b_2) \\ 0 & 0 & 0 & -2(3b_1+b_3) \end{bmatrix} \Rightarrow -2(3b_1+b_3)=0$$

$$3b_1+b_3=0$$

Computing $A\bar{x}$

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & -1 & 4 \\ -2 & 7 & -5 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

way 1:

$$Ax = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} x_1 + \begin{bmatrix} -2 \\ -1 \\ 7 \end{bmatrix} x_2 + \begin{bmatrix} 4 \\ 4 \\ -5 \end{bmatrix} x_3$$

$$= \begin{bmatrix} x_1 \\ 3x_1 \\ -2x_1 \end{bmatrix} + \begin{bmatrix} -2x_2 \\ -x_2 \\ 7x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ 4x_3 \\ -5x_3 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 4x_3 \\ 3x_1 - x_2 + 4x_3 \\ -2x_1 + 7x_2 - 5x_3 \end{bmatrix}$$

way 2:

$$Ax = \begin{bmatrix} 1 & -2 & 4 \\ 3 & -1 & 4 \\ -2 & 7 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - 2x_2 + 4x_3 \\ 3x_1 - x_2 + 4x_3 \\ -2x_1 + 7x_2 - 5x_3 \end{bmatrix} \qquad \begin{matrix} R_1 \cdot \bar{x} \\ R_2 \cdot \bar{x} \\ R_3 \cdot \bar{x} \end{matrix}$$

Examples:

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} (2-5) \\ (4+5) \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 1 \\ -1 & 7 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} (8+0+2) \\ (-2-7+6) \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$$

Properties of Matrix-Vector multiplication

A is $m \times n$ matrix
u & v vectors in $\mathbb{R}^n$
c is a scaler

1. $A(\bar{u}+\bar{v}) = A\bar{u} + A\bar{v}$

2. $A(c\bar{u}) = c(A\bar{u})$

Note: $A\bar{x}$
$(m \times n)(n \times 1)$
must be same