1.5: Solution Sets of Linear Systems

Homogeneous Linear Systems

Homogeneous Linear systems : $A\bar{x} = \bar{0}$

* Always has at least 1 solution : $\bar{x} = \bar{0}$

trivial solution : $\bar{x} = 0$

nontrivial solutions : $\bar{x} \neq 0$

System $Ax = 0$ has a non-trivial solution $\Leftrightarrow$ the equation has at least 1 variable.

ex :

$$\begin{cases} 2x_1 + 4x_2 + 3x_3 = 0 \\ -x_1 + 5x_2 + 3x_3 = 0 \\ -7x_1 + 7x_2 + 3x_3 = 0 \end{cases}$$

$$\begin{bmatrix} 2 & 4 & 3 & 0 \\ -1 & 5 & 3 & 0 \\ -7 & 7 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 5 & 3 & 0 \\ 2 & 4 & 3 & 0 \\ -7 & 7 & 3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} +1 & -5 & -3 & 0 \\ 0 & 14 & 9 & 0 \\ 0 & -28 & -18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -3 & 0 \\ 0 & 14 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -5 & -3 & 0 \\ 0 & 1 & 9/14 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3/14 & 0 \\ 0 & 1 & 9/14 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 3/14 & 0 \\ 0 & 1 & 9/14 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

means:

$$\begin{aligned} x_1 + \tfrac{3}{14} x_3 &= 0 \\ x_2 + \tfrac{9}{14} x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

Answer in parametric form:

$$\begin{aligned} x_1 &= -\tfrac{3}{14} x_3 \\ x_2 &= -\tfrac{9}{14} x_3 \\ x_3 &= x_3 \quad \text{or free variable} \end{aligned}$$

Answer in vector form:

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\tfrac{3}{14} x_3 \\ -\tfrac{9}{14} x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\tfrac{3}{14} \\ -\tfrac{9}{14} \\ 1 \end{bmatrix} x_3$$

ex: ~~the solution set of a linear system~~
Describe all the solutions of the linear system:

$$2x_1 - 3x_2 - 5x_3 = 0$$

solve for $x_1 = \tfrac{3}{2} x_2 + \tfrac{5}{2} x_3$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \tfrac{3}{2} x_2 + \tfrac{5}{2} x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \tfrac{3}{2} x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \tfrac{5}{2} x_3 \\ 0 \\ x_3 \end{bmatrix}$$

Parametric vector form $\longrightarrow$ $= x_2 \begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5/2 \\ 0 \\ 1 \end{bmatrix}$

Parametric Vector Form

$$\bar{X} = \bar{u}_1 x_1 + \bar{u}_2 x_2 + \cdots + \bar{u}_{n-1} x_{n-1} + \bar{u}_{n}$$

some vector — our variables — can only be nonzero if we have $\bar{b} \neq 0$

example:

$$\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} x_2$$

Nonhomogeneous Linear Systems

$$A\bar{x} = \bar{b}, \quad \text{where } \bar{b} \neq 0$$

ex:

$$\begin{cases} x_1 + 3x_2 - 5x_3 = 4 \\ x_1 + 4x_2 - 8x_3 = 7 \\ -3x_1 - 7x_2 + 9x_3 = -6 \end{cases}$$

find solution in parametric vector form:

$$\begin{bmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{matrix} x_1 + 4x_3 = -5 \\ x_2 - 3x_3 = 3 \\ 0 = 0 \end{matrix} \quad \Rightarrow \quad \begin{cases} x_1 = -4x_3 - 5 \\ x_2 = 3x_3 + 3 \end{cases}$$

regular parametric:

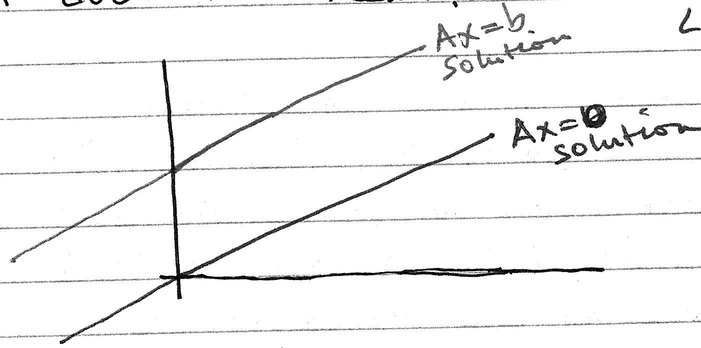
$$\begin{cases} x_1 = -4x_3 - 5 \\ x_2 = 3x_3 + 3 \\ x_3 \text{ free} \end{cases}$$

shifted the answer

vector parametric:

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_3 - 5 \\ 3x_3 + 3 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}}_{\substack{\text{general solution associated} \\ \text{to the homogeneous linear system}}} x_3 + \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$$

what does this mean?



~~they're parallel~~

they're parallel lines/planes/etc

Thm:

$A\bar{x} = \bar{b}$ is consistent for some $\bar{b}$ and $\bar{p}$ is a solution of $A\bar{x} = \bar{b}$ then the solution set is of the form $w = p + v_h$

where v_h is a solution of $A\bar{x} = \bar{0}$

Result: if $A\bar{x} = \bar{0} \Rightarrow \bar{x} = 0$ then solution of $A\bar{x} = \bar{b}$ is unique

Method to get solution set in Parametric Vector form:

1. row reduce the augment matrix to reduced echelon form

2. Express each variable in terms of any free variables.

——— At this point we have our solution set in parametric form

3. Write a typical solution $\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ as a vector whose entries are in terms of the free variables (if no free variables, then unique solution)

4. Decompose $\bar{x}$ into a Linear combination of vectors (w/ # entries) using the free variables as parameters. (ie: the vectors will be the coef of a particular free variable)

1.6: Applications of Linear Systems

Economics: Exchange Model

Economy divided into sectors

for each we know:
- total output for 1 year
- how output divided (or exchanged) among the other sectors

Price of output = total \$\$ value of output

Then exists equilibrium prices that can be assigned to the total outputs of each sector ∋ the income balences the expenses

ex: economy: Coal, Electric, Steel

Output: Coal	Electric	Steel	Purchased by
0	.4	.6	Coal
.6	.1 Expense to Elec	.2	Electric
.4	.5	.2 Expense to Steel	Steel

input needs →

↑ how output divided

Find the prices (in \$\$) of the total annual outputs of Coal, Electric, & Steel by P_C, P_E, P_S. If possible find equilibrium prices

Coal: $\underbrace{P_C}_{\text{outputs}} = \overbrace{.4P_E + .6P_S}^{\text{inputs}}$ (from row 1)

Elect: $P_E = .6P_C + .1P_E + .2P_S$ (from row 2)

Steel: $P_S = .4P_C + .5P_E + .2P_S$ (from row 3)

now the problem is in math terms.
we put it in std format for a Linear System:

$$P_c - .4P_E - .6P_S = 0$$
$$-.6P_c + .9P_E - .2P_S = 0$$
$$-.4P_c - .5P_E + .8P_S = 0$$

$$\left[\begin{array}{cccc} 1 & -.4 & -.6 & 0 \\ -.6 & .9 & -.2 & 0 \\ -.4 & -.5 & .8 & 0 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & -.4 & -.6 & 0 \\ -.6 & .9 & -.2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc} 1 & -.4 & -.6 & 0 \\ 0 & .66 & -.56 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & -.4 & -.6 & 0 \\ 0 & 1 & -.\overline{84} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad -.\overline{84} = \frac{-28}{33} \quad -.\overline{93} = \frac{-31}{33}$$

$$\sim \left[\begin{array}{cccc} 1 & 0 & -.\overline{93} & 0 \\ 0 & 1 & -.\overline{84} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} P_c = .\overline{93}P_S \\ P_E = .\overline{84}P_S \\ 0 = 0 \end{array}$$

price vector

$$\bar{P} = \begin{bmatrix} P_c \\ P_E \\ P_S \end{bmatrix} = \begin{bmatrix} .\overline{93}P_S \\ .\overline{84}P_S \\ P_S \end{bmatrix} \approx \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix} P_S$$

all positive means $\bar{P}$ is our equilibrium price vector

Network Flow

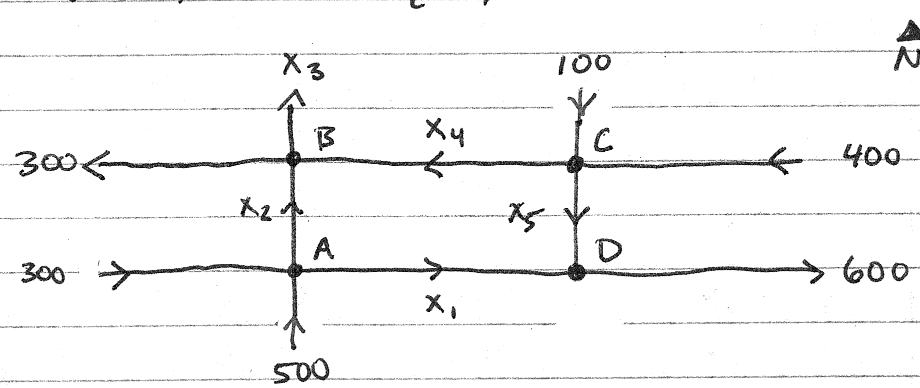
Network – contains nodes & braches

Assumption: inflow = outflow
(ie: no deadends or sources in the network)

* Conservation: inflow = outflow
- at each node/junction
- over entire network

Example: traffic patterns

Determine the general flow pattern
(ie: find all x_i's)



Intersection	inflow		outflow
A	$300 + 500$	=	$x_1 + x_2$
B	$x_2 + x_4$	=	$300 + x_3$
C	$400 + 100$	=	$x_4 + x_5$
D	$x_1 + x_5$	=	600

$$\left.\begin{array}{ll} \text{total in:} & 300 + 100 + 400 + 500 \\ \text{total out:} & 300 + x_3 + 600 \end{array}\right\} \Rightarrow x_3 = 400$$

Put Linear System in std form:

$$\begin{cases} X_1 + X_2 = 800 & (A) \\ X_2 - X_3 + X_4 = 300 & (B) \\ X_4 + X_5 = 500 & (C) \\ X_1 + X_5 = 600 & (D) \\ X_3 = 400 & (\text{total}) \end{cases}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 0 & 1 & 0 & 0 & 400 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 600 \\ 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & 1 & -1 & 1 & 0 & 300 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & 1 & -1 & 1 & 0 & 300 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & -1 & 1 & 1 & 100 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 1 & 0 & 0 & -1 & 200 \\ 0 & 0 & 1 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} X_1 + X_5 = 600 \\ X_2 - X_5 = 200 \\ X_3 = 400 \\ X_4 + X_5 = 500 \end{array}$$

$$\begin{cases} X_1 = 600 - X_5 \\ X_2 = 200 - X_5 \\ X_3 = 400 \\ X_4 = 500 - X_5 \\ X_5 = \text{free} \end{cases}$$