1.7: Linear Independence

Linearly Independent

set $\{\bar{v}_1, \dots, \bar{v}_p\}$ in $\mathbb{R}^n$ are Linearly independent
if $\quad c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_p\bar{v}_p = \bar{0}$
is true only if $c_1 = c_2 = \dots = c_p = 0$
(ie: only has the trivial solution)

Linearly Dependent

set $\{\bar{v}_1, \dots, \bar{v}_p\}$ in $\mathbb{R}^n$ are Linearly dependent
if $\quad c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_p\bar{v}_p = \bar{0}$
then $\exists\ c_1, c_2, \dots, c_p$ where not all $c_i = 0$

<u>example</u>:

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -3 \\ -1 \\ 6 \end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

a) determine if the set is Linearly independent
b) if possible, find a linear dependence relation among v_1, v_2, v_3 (ie: find c_1, c_2, c_3)

Solution:

$$c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{0}$$

$$c_1\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} -3 \\ -1 \\ 6 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} -3c_2 + c_3 \\ c_1 - c_2 + c_3 \\ 3c_1 + 6c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

<u>OR</u> Augmented matrix

$$\begin{bmatrix} 0 & -3 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -3 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 9 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2/3 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{aligned} c_1 &= -2/3\, c_3 \\ c_2 &= +1/3\, c_3 \\ c_3 &= \text{free} \end{aligned}$$

pick $c_3 = 3 \Rightarrow c_2 = 1$, $c_1 = -2$

$\Longrightarrow -2\bar{v}_1 + \bar{v}_2 + 3\bar{v}_3 = \bar{0}$ dependent

Linear Independence of Matrix Columns

the Columns of a matrix A are Linearly independent iff $A\bar{x} = \bar{0}$ only has the trivial solution

1 or 2 Vectors

1 vector : Linearly independent if not zero vector

2 vectors: Linearly dependent if 1 vector is a multiple of the other

examples:

a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \end{bmatrix}$ b) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ d) $\begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$

2+ vectors

the set is linearly dependent if at least 1 of the vectors is a linear combination of the others

Why?

$$c_1 \bar{v}_1 + c_2 \bar{v}_2 + \dots + c_p \bar{v}_p = \bar{0}$$

if $c_1 \neq 0$

$$\Rightarrow \quad -c_1 \bar{v}_1 = c_2 \bar{v}_2 + \dots + c_p \bar{v}_p$$

$$\bar{v}_1 = -\frac{c_2}{c_1} \bar{v}_2 + \dots + -\frac{c_p}{c_1} \bar{v}_p$$

<u>Thrm</u>: If a set of vectors contains more vectors than entries in each vector, then the set is linearly dependent
(ie: $\{\bar{v}_1, \dots, \bar{v}_p\}$ in $\mathbb{R}^n$ dependent if $p > n$)

<u>Examples</u>:

If a set of vectors $S = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

examples:

a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

b) $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

c) $\begin{bmatrix} 2 \\ 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ 5 \\ -15 \\ -15 \end{bmatrix}$

d) $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

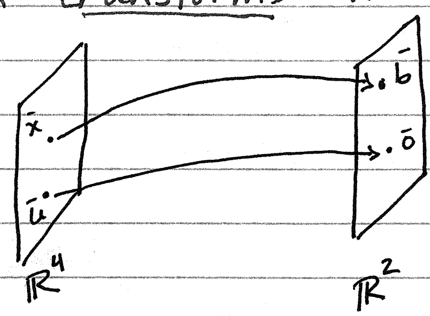
1.8: Intro to Linear Transformations

Consider: $Ax = b$

$$\underbrace{\begin{bmatrix} 1 & 3 & -5 & 1 \\ 1 & 4 & 0 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 7 \end{bmatrix}}_{b} \quad \& \quad \underbrace{\begin{bmatrix} 1 & 3 & -5 & 1 \\ 1 & 4 & 0 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 2 \\ 4 \\ 1 \\ -9 \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{0}$$

- can think of A as "acting" on vector x to get us a new vector Ax

- A <u>transforms</u> $\bar{x}$ into $\bar{b}$ and $\bar{u}$ into $\bar{0}$



means solving $Ax = b$ is really finding all x in $\mathbb{R}^4$ such that A transforms these vectors into $\bar{b}$

Transformation T

a function or mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns each vector $\bar{x}$ in $\mathbb{R}^n$ to a vector $T(\bar{x})$ in $\mathbb{R}^m$

domain of T is $\mathbb{R}^n$

codomain of T is $\mathbb{R}^m$

$T: \mathbb{R}^n \to \mathbb{R}^m$ read as T maps $\mathbb{R}^n$ to $\mathbb{R}^m$