1.9: the matrix of a Linear Transformation

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ or $\bar{x} \longmapsto A\bar{x}$

is completely determined by what does to the columns of the $n \times n$ identity matrix I_n

<u>ex</u>:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that

$$T(e_1) = \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix} \quad \text{and} \quad T(e_2) = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}$$

Find $T(\bar{x})$ and $A \ni T(\bar{x}) = A\bar{x}$

step 1 :

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2 = x_1 e_1 + x_2 e_2$$

step 2 : T Linear so

$$T(\bar{x}) = T(x_1\bar{e}_1 + x_2\bar{e}_2) = x_1 T(\bar{e}_1) + x_2 T(\bar{e}_2)$$

$$= x_1 \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 5x_2 \\ -x_1 \\ 7x_1 + 3x_2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 2 & 5 \\ -1 & 0 \\ 7 & 3 \end{bmatrix}}_{\Uparrow} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = [T(e_1) \;\; T(e_2)]$$

Thrm:

$T: \mathbb{R}^n \to \mathbb{R}^m$ linear transformation
then $\exists!$ matrix $A \ni T(\bar{x}) = A\bar{x} \quad \forall \bar{x} \in \mathbb{R}^n$
and A is $m \times n$ where

$$A = [T(e_1) \quad T(e_2) \quad \dots \quad T(e_n)]$$

the standard matrix for linear transformation T:
A as def above

ex:

Find the stand matrix for $T(\bar{x}) = 3\bar{x}$, $\bar{x} \in \mathbb{R}^3$

step 1:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

step 2:

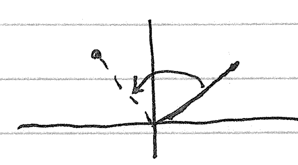
$$A = [T(e_1) \quad T(e_2) \quad T(e_3)]$$

so
$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

ex: Rotations

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$
rotates each pt in $\mathbb{R}^2$ about the origin
widdershins for pos angle φ and
clockwise for neg angle φ
Find the standard matrix A

$$A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$



* Pictures of reflections (p85) thru (p87)
& Contractions, shears, projections.

Def: Onto

A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if each $\bar{b}$ in $\mathbb{R}^m$ is the image of at least 1 $\bar{x}$ in $\mathbb{R}^n$

(ie: for each $\bar{b}$ in $\mathbb{R}^m$, you can find an $\bar{x}$ in $\mathbb{R}^n$ such that $T(\bar{x}) = \bar{b}$)

Def: One-to-one

A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is 1-1 if each $\bar{b}$ in $\mathbb{R}^m$ is the image of at most one $\bar{x}$ in $\mathbb{R}^n$

(ie: if $T(\bar{x}) = \bar{b}$ then there are no other $\bar{x}$'s, call it $\bar{y}$, such that $T(\bar{y}) = \bar{b}$)

Uniqueness

if $T(\bar{x}) = A\bar{x}$ is 1-1

then $A\bar{x} = b$ has exactly 1 solution (unique) or no solutions

Not 1-1 means we can have multiple solutions

Thrm

$T: \mathbb{R}^n \to \mathbb{R}^m$ Linear transformation

then T is 1-1 $\Longleftrightarrow$ $T(x) = 0$ has only the trivial solution.

Why?

proof:

T linear $\Rightarrow$ $T(0) = 0$

$(\Rightarrow)$ if T is 1-1 $\Rightarrow$ $T(\bar{x}) = 0$ has at most one solution $\Rightarrow$ $\bar{x} = 0$

$(\Leftarrow)$ if T is not 1-1 then $\exists$ a $\bar{b}$ where both $\bar{u}$ & $\bar{v}$ give $T(\bar{u}) = \bar{b}$ & $T(\bar{v}) = \bar{b}$

T linear:

$T(u-v) = T(u) - T(v) = b - b = 0$

and $u - v \neq 0$ b/c $u \neq v$

so $T(x) = 0$ has more than 1 solution

but the assumption was that $T(x) = 0$ only had the trivial solution

Thrm:

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear transformation and A be the std matrix for T then:

a) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ $\Leftrightarrow$ the columns of A span $\mathbb{R}^m$

b) T is 1-1 $\Leftrightarrow$ the columns of A are linearly independent

means/results:

a) Onto

Col(A) span $\mathbb{R}^m$ iff # of columns is m or more

b) 1-1

Col(A) are linearly independent iff # of columns is m or less

When both? ANS: square matrix

examples:

Suppose $T(\bar{x}) = A\bar{x}$, which A's will give us the possibility of onto or 1-1 (ie: when not onto or 1-1 by just looking)

a) $\begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 5 & 3 \end{bmatrix}$ can't span so not onto

b) $\begin{bmatrix} -4 & 7 & -8 & 2 \\ 1 & -1 & 0 & 1 \\ -5 & 0 & 3 & 0 \end{bmatrix}$ not 1-1

c) $\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ ~~spans~~ onto & 1-1

d $\begin{bmatrix} 2 & 5 & -1 \\ -1 & -3 & 0 \end{bmatrix}$ not 1-1

ex:

Show $T: \mathbb{R}^2 \to \mathbb{R}^3$ is 1-1

where $T(x_1, x_2) = (2x_1 - x_2, x_1, x_1 + 5x_2)$

1 way: we put into column vectors:

$$T(\bar{x}) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 \\ x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

real question: are $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix}$ linearly independent?

$$c_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 5 & 0 \end{bmatrix} \text{ and rref}$$

OR

$$\begin{cases} 2c_1 - c_2 = 0 \\ c_1 = 0 \\ c_1 + 5c_2 = 0 \end{cases} \Rightarrow \begin{cases} 2\cdot 0 - c_2 = 0 \\ c_1 = 0 \\ 0 + 5c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$$

Linearly independent

So, yes T is 1-1

ex 1: Constructing a Nutritious Weight-loss Diet

Research behind the 1980's Cambridge Diet
below is a simple illustration:

Nutrient	Amounts (g) Supplied per 100g of ingredient: non-fat Milk	Soy flour	Whey	Cambridge
Protein	36	51	13	33
Carbs	52	34	74	45
Fat	0	7	1.1	3

Find some combination of milk, soy, & whey to provide the exact amounts of protein, carb, fat supplied by the diet in 1 day.

x_1 - # of units of Milk we want (in 100g)
x_2 - " Soy "
x_3 - " Whey "

$$\left.\begin{aligned} \text{Protein: } & 36x_1 + 51x_2 + 13x_3 = 33 \\ \text{Carbs: } & 52x_1 + 34x_2 + 74x_3 = 45 \\ \text{fat: } & 0\cdot x_1 + 7x_2 + 1.1x_3 = 3 \end{aligned}\right\} \text{our linear system to solve}$$

$$\Rightarrow \begin{bmatrix} 36 & 51 & 13 & 33 \\ 52 & 34 & 74 & 45 \\ 0 & 7 & 1.1 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & .2772 \\ 0 & 1 & 0 & .3919 \\ 0 & 0 & 1 & .2332 \end{bmatrix}$$

means: we need .2772 units of milk (277.2 g), .3919 units of soy (391.9 g), and .2332 units of whey (233.2 g) per day to satisfy the same amounts as the diet

* important *

needed $x_1, x_2, \& x_3$ to be non-negative

how could we have used $-.5$ units of milk? we couldn't.

Notice we sometimes have constraints on the problem beyond the linear equations.

Ex 2: Electrical Networks

Voltage Source ⇒ ie: batteries
this is where the electricity (ie: our electrons) comes "from"

Resistor ⇒ ie: appliances
this is something that "uses up" the electricity

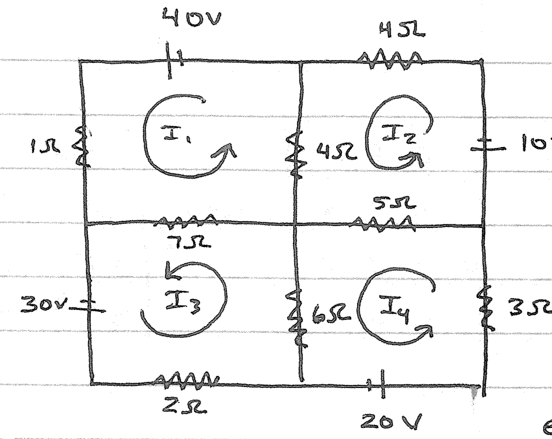
Ohm's Law: $V = RI$

V = voltage in volts (v)

R = resistence in Ohms (Ω)

I = current in amps

An electrical network:



exercise #7 from ~~text~~

- designations of loop currents are arbitrary (ie: the direction of the arrows is arbitrary)
  - arrow from pos to neg, means pos I ; arrow from neg to pos, means I neg
- Voltage Source ⊣⊢
  longer side = positive
  shorter side = neg

> Kirchhoff's Voltage Law:
> Algebraic sum of the RI voltage drops around a loop in 1 direction
> =
> Algebraic sum of the voltage sources in the same direction around the loop

Determine the loop currents in our network

loop 1: $I_1 + 7I_1 + 4I_1 = 12I_1$ (inside loop)

$-7I_3$ & $-4I_2$ (outside loop)

$$12I_1 - 4I_2 - 7I_3 = 40$$

↑ volt source

loop 2: $-4I_1 + 13I_2 - 5I_4 = -10$

Loop 3: $-7I_1 + 15I_3 - 6I_4 = 30$

Loop 4: $-5I_2 - 6I_3 + 14I_4 = 20$

Solving:

$$\begin{bmatrix} 12 & -4 & -7 & 0 & 40 \\ -4 & 13 & 0 & -5 & -10 \\ -7 & 0 & 15 & -6 & 30 \\ 0 & -5 & -6 & 14 & 20 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 0 & 11.434 \\ 0 & 1 & 0 & 0 & 5.8396 \\ 0 & 0 & 1 & 0 & 10.55 \\ 0 & 0 & 0 & 1 & 8.0357 \end{bmatrix}$$

So
$I_1 = 11.434$
$I_2 = 5.8396$
$I_3 = 10.55$
$I_4 = 8.0357$

(+) current means the arrows are drawn in the same direction as the current (ie: we drew the picture correctly)