Definition: Matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

row i

column j
($\bar{a}_j$)

diagonal entries: $a_{11}, a_{22}, \ldots$
they are the <u>main</u> diagonal

diagonal matrix: square matrix
where the non-diagonal entries are zero
<u>ex</u>: I_n

Zero matrix: all entries are zero

equal $A = B$ if same size & $a_{ij} = b_{ij}$ for all i & j

sum $A+B$ add the corresponding entries

scaler multiple rA multiply each entry of A by r

transpose A^T interchange the rows/columns

<u>ex</u>: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

Properties of Addition & Scaler Multiplication

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$A + B = B + A$ $\quad$ $r(A+B) = rA + rB$

$(A+B)+C = A + (B+C)$ $\quad$ $(r+s)A = rA + sA$

$A + 0 = A$ $\quad$ $r(sA) = (rs)A$

Properties of Matrix Multiplication

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$A(BC) = (AB)C$ associative law

$A(B+C) = AB + AC$ left distributive law

$(B+C)A = BA + CA$ right distributive law

$r(AB) = (rA)B = A(rB)$ for any scaler r

$I_m A = A = A I_n$

where A is $m \times n$ matrix

* in general $AB \neq BA$ *

– different sizes

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 17 & 22 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -4 \\ 2 & 5 & 9 \\ 8 & 17 & 24 \end{bmatrix}$$

– $$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 5 \end{bmatrix}$$

just b/c

* if $AB = AC \not\Rightarrow B = C$

* if $AB = 0 \not\Rightarrow A = 0$ or $B = 0$

Properties of Transpose

$$(A^T)^T = A \qquad (rA)^T = rA^T \;, \quad r \text{ scaler}$$

$$(A+B)^T = A^T + B^T \qquad (AB)^T = B^T A^T$$

Examples

$$\begin{bmatrix} 4 & 5 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 1 & -3 \end{bmatrix}$$

$$3\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -3 \\ 0 & 6 & 12 \\ 0 & 0 & 3 \end{bmatrix}$$

$$2\begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 0 & -4 \end{bmatrix} - 4\begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix} = \text{can't compute}$$

$$\begin{bmatrix} 2 & 4 \\ -1 & 0 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 3 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \\ -4 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & -4 \\ 2 & 1 & 0 \end{bmatrix}$$

Matrix Multiplication

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \cdot \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$

(3×4) (4×5) (3×5)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 4 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 6 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

↑ rows (2×2) ↑ columns (2×3)

a: $1 \cdot 2 + 2 \cdot -1$ d: $3 \cdot 2 + 4 \cdot -1$

b: $1 \cdot 0 + 2 \cdot 4$ e: $3 \cdot 0 + 4 \cdot 4$

c: $1 \cdot 6 + 2 \cdot 1$ f: $3 \cdot 6 + 4 \cdot 1$

$$AB = \begin{bmatrix} 0 & 8 & 8 \\ 2 & 16 & 22 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 \\ 2 & -4 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \\ 0 & -4 & 1 \end{bmatrix} = \begin{bmatrix} (2 \cdot -1 + 0) & (4 - 20) & (2 + 5) \\ (-2 + 0) & (4 + 16) & (2 - 4) \\ (-7 + 0) & (14 - 4) & (7 + 1) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -16 & 7 \\ -2 & 20 & -2 \\ -7 & 10 & 8 \end{bmatrix}$$

2.2: the inverse of a matrix

Inverses:

previously: x^{-1} , 2^{-1} , $.4^{-1}$

means: $x \cdot x^{-1} = 1$, $2 \cdot 2^{-1} = 1$, $.4^{-1} \cdot .4 = 1$

Matrix Inverses

→ Matrix must be square $(n \times n)$

if C is the inverse of A, then both

$$CA = I_n \quad \text{and} \quad AC = I_n$$

we call $C = A^{-1}$

$$A^{-1}A = I = AA^{-1}$$

ex: $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

$$AC = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$CA = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2x2 Inverses

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if $ad - bc \neq 0$ then A invertible

and $A^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Def: $\det(A) = ad - bc$ if A is 2×2

so A invertible if and only if $\det(A) \neq 0$

Def: a singular matrix has a determinent which equals zero

Fact: nonsingular matrix = invertible matrix

ex: find A^{-1} if $A = \begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix}$

$$A^{-1} = \frac{1}{2\cdot 4 - 0\cdot -3}\begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 3/8 \\ 0 & 1/4 \end{bmatrix}$$

Properties of Inverses

1. $A\bar{x} = \bar{b}$ and A^{-1} exists then $\bar{x} = A^{-1}\bar{b}$ is unique & exists $\forall \bar{b}$

2. $(A^{-1})^{-1} = A$

3. $(AB)^{-1} = B^{-1}A^{-1}$

4. $(A^{T})^{-1} = (A^{-1})^{T}$

Algorithm for finding A^{-1}:

(ie: how to do it & the steps to take)

To find A^{-1}:

1) start with A an $n \times n$ matrix

2) form the matrix $[A \,\vdots\, I]$
   where I is the $n \times n$ identity matrix

3) rref $[A \,\vdots\, I]$

4) if $[A \,\vdots\, I] \rightsquigarrow [I \,\vdots\, A^{-1}]$
   you found A^{-1}

   if $[A \,\vdots\, I] \rightsquigarrow [J \,\vdots\, B]$
   where $I \neq J$
   then A doesn't have an inverse.

example: find A^{-1} if $A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 0 \\ -2 & 1 & 3 \end{bmatrix}$

$[A \,\vdots\, I]$

$$= \left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ -2 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \\ R_2 \leftrightarrow \frac{1}{4} R_2 \\ R_3 \leftrightarrow R_3 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/4 & 0 \\ 0 & 7 & 13 & 2 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \leftrightarrow R_1 - 3R_2 \\ \\ R_3 \leftrightarrow R_3 - 7R_2 \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 5 & 1 & -\frac{3}{4} & 0 \\ 0 & 1 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 13 & 2 & -7/4 & 1 \end{array}\right] \quad R_3 \leftrightarrow R_3 \cdot \frac{1}{13}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 5 & 1 & -3/4 & 0 \\ 0 & 1 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & 2/13 & -7/52 & 1/13 \end{array}\right] \quad R_1 \leftrightarrow R_1 - 5R_3$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3/13 & -1/13 & -5/13 \\ 0 & 1 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & 2/13 & -7/52 & 1/13 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} 3/13 & -1/13 & -5/13 \\ 0 & 1/4 & 0 \\ 2/13 & -7/52 & 1/13 \end{bmatrix}$

Check it:

$$AA^{-1} = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 0 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3/13 & -1/13 & -5/13 \\ 0 & 1/4 & 0 \\ 2/13 & -7/52 & 1/13 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{3}{13}+\frac{10}{13}\right) & \left(-\frac{1}{13}+\frac{3}{4}-\frac{35}{52}\right) & \left(-\frac{5}{13}+0+\frac{5}{13}\right) \\ 0 & 1 & 0 \\ \left(-6/13+6/13\right) & \left(\frac{2}{13}+\frac{1}{4}-\frac{21}{52}\right) & \left(\frac{10}{13}+\frac{3}{13}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 3/13 & -1/13 & -5/13 \\ 0 & 1/4 & 0 \\ 2/13 & -7/52 & 1/13 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 0 \\ -2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{3}{13}+\frac{10}{13}\right) & \left(\frac{9}{13}-\frac{4}{13}-\frac{5}{13}\right) & \left(\frac{15}{13}-\frac{15}{13}\right) \\ 0 & 1 & 0 \\ \left(\frac{2}{13}-\frac{2}{13}\right) & \left(\frac{6}{13}-\frac{7}{13}+\frac{1}{13}\right) & \left(\frac{10}{13}+\frac{3}{13}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$