# 2.5: LU Factorization

Factorization:

start w/ matrix. find 2 other matrices that when ~~add~~ multiplied together equal the 1st matrix.

LU Factorization:

$$A\bar{x} = \bar{b}$$

- have matrix A
- want $L \cdot U = A$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \cdot \begin{bmatrix} \square & * & * & * & * \\ 0 & \square & * & * & * \\ 0 & 0 & 0 & \square & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = L \cdot U$$

L = unit lower triangular

U = upper triangular

Why would we want to do this?

Solving $A\bar{x} = \bar{b}$ by using $A^{-1}$ takes $2n^3$ operations

Solving $A\bar{x} = \bar{b}$ by LU factorization takes $\frac{2}{3}n^3$ operations

* divides time by 3 *

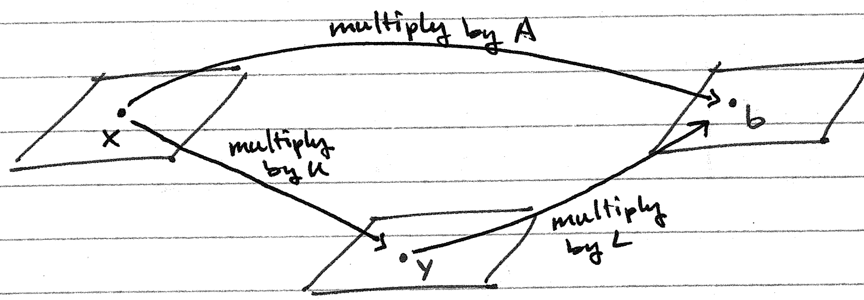
## What's Going On?

$A\bar{x} = b$ and $A = LU$

$L(Ux) = b$ $\qquad y = Ux$

$Ly = b$

Steps: solve $Ly = b$ for $y$
Solve $Ux = y$ for $x$




ex:

$$A = \begin{bmatrix} -1 & 2 & 4 & 1 \\ -4 & 15 & 13 & 6 \\ 1 & 12 & -8 & 2 \\ -3 & -29 & 31 & -6 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 3 & -5 & 2 & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} -1 & 2 & 4 & 1 \\ 0 & 7 & -3 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}}_{U}$$

a) Verify this factorization

b) Solve $A\bar{x} = \bar{b}$, where $b = \begin{bmatrix} 9 \\ 35 \\ -6 \\ 39 \end{bmatrix}$

step 1: solve $Ly=b$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 9 \\ 4 & 1 & 0 & 0 & 35 \\ -1 & 2 & 1 & 0 & -6 \\ 3 & -5 & 2 & 1 & 39 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right]$$

$\bar{y}$

step 2: solve $Ux=y$

$$\left[\begin{array}{cccc|c} -1 & 2 & 4 & 1 & 9 \\ 0 & 7 & -3 & 2 & -1 \\ 0 & 0 & 2 & -1 & 5 \\ 0 & 0 & 0 & 3 & -3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$\bar{x}$

Answer: $\bar{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}$

## Finding the LU Factorization

1. Reduce A to an Echelon form U by a sequence of row operations

2. Place Entries in L $\ni$ the same sequence of row operations will reduce L to I.

But how do we really do that?

ex: find L & U where $A = L \cdot U$

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

finding U:

$$A = \begin{bmatrix} \boxed{2} & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & \boxed{3} & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & \boxed{2} & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U$$

making L:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & & 1 & 0 \\ -3 & & & 1 \end{bmatrix}$$

L always square & has 1's on diagonal

$2 \longrightarrow 1$ by $\div 2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & & 1 \end{bmatrix}$$

$3 \longrightarrow 1$ by $\div 3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix} = L$$

$2 \longrightarrow 1$ by $\div 2$

U - upper $\Delta$

L - lower unit $\Delta$

* Always check: $A = L \cdot U$

ex: Find $L$ & $U$ such that $A = LU$

$$A = \begin{bmatrix} 1 & 3 & 4 & 0 \\ -3 & -6 & -7 & 2 \\ 3 & 3 & 0 & -4 \\ -5 & -3 & 2 & 9 \end{bmatrix}$$

finding $U$: making $L$:

$$A = \begin{bmatrix} \boxed{1} & 3 & 4 & 0 \\ -3 & -6 & -7 & 2 \\ 3 & 3 & 0 & -4 \\ -5 & -3 & 2 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & & 1 & 0 \\ -5 & & & 1 \end{bmatrix} \quad \begin{matrix} 1 \to 1 \\ \text{no change} \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & \boxed{3} & 5 & 2 \\ 0 & -6 & -12 & -4 \\ 0 & 12 & 22 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & & 1 \end{bmatrix} \quad \begin{matrix} 3 \to 1 \\ \div 3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & \boxed{-2} & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \qquad \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -1 & 1 \end{bmatrix}}_{=L} \quad \begin{matrix} -2 \to 1 \\ \div -2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} = U$$

Check $A = L \cdot U$ ✓