2.6: the Leontief Input-Output Model

An Economic Model

total economy = n parts
(ie: produce cars
produce clothes, etc)

$\bar{x}$ = production vector in $\mathbb{R}^n$
gives output of each part of economy for 1 year

$\bar{d}$ = final demand vector
lists value of goods & the services demanded by the non-producing part of the economy (ie: people not making cars want them) This can be considered the consumer demand

intermediate demand = what the producer needs in order to make their product

want formula:

$$\left\{\begin{matrix}\text{amount}\\ \text{produced}\\ \bar{x}\end{matrix}\right\} = \left\{\begin{matrix}\text{intermediate}\\ \text{demand}\end{matrix}\right\} + \left\{\begin{matrix}\text{final}\\ \text{demand}\\ \bar{d}\end{matrix}\right\}$$

the Leontief Input-Output Model:

$$\bar{x} = C\bar{x} + \bar{d}$$

amount produced

consumption matrix

final demand

Consumption Matrix:

* rewrite data chart as matrix *
all numbers must be in decimal (ie: %)

ex:

Inputs Consumed per Unit Output

perchased from:	Manuf	Agri	Serv
Manufacturing	.5	.4	.2
Agriculture	.2	.3	.1
Services	.1	.1	.3
	c_1	c_2	c_3

so *Manufacturing needs*
.1 from manufacturing
.4 from Agriculture
& .2 from services

Consumption Matrix:

$$C = \begin{bmatrix} .5 & .4 & .2 \\ .2 & .3 & .1 \\ .1 & .1 & .3 \end{bmatrix}$$

ex: Suppose we have the same economy as before. If the final demand is 50 units of manufacturing, 30 units for agriculture, and 20 units for services, find the production level $\bar{x}$ needed to satisfy this demand.

$$\bar{x} = C\bar{x} + \bar{d}$$

$$\bar{x} - C\bar{x} = \bar{d}$$
$$(I - C)\bar{x} = \bar{d}$$

$$\bar{x} = (I - C)^{-1}\bar{d}$$

$$\bar{x} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .5 & .4 & .2 \\ .2 & .3 & .1 \\ .1 & .1 & .3 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 50 \\ 30 \\ 20 \end{bmatrix}$$

$$\approx \begin{bmatrix} 226 \\ 119 \\ 78 \end{bmatrix}$$ rounded to integers

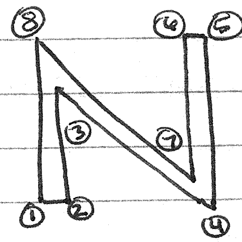
So, we need to produce
226 units of manufacturing
119 units of agriculture
78 units of services

Thm:

For the Leontief model $\bar{x} = C\bar{x} + d$ has a unique solution (with non-negative #s) if $(I - C)^{-1}$ exists

meaning C & d have non-neg entries & if each column sum of C is less than 1 types of transformations :

ex: the letter N



$$N = \begin{array}{c} \\ x\text{'s} \\ y\text{'s} \end{array} \begin{array}{cccccccc} ① & ② & ③ & ④ & ⑤ & ⑥ & ⑦ & ⑧ \\ \end{array}$$

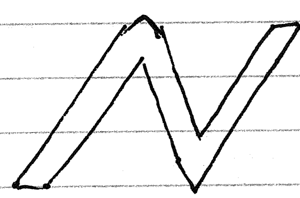
$$\begin{bmatrix} 0 & .5 & .5 & 6 & 6 & 5.5 & 5.5 & 0 \\ 0 & 0 & 6.42 & 0 & 8 & 8 & 1.58 & 8 \end{bmatrix}$$

a Shear transformation : $A = \begin{bmatrix} 1 & .25 \\ 0 & 1 \end{bmatrix}$

multiply AN will get matrix of the new vertices :

$$\begin{array}{cccccccc} ① & ② & & ④ & & & & \end{array}$$

$$AN = \begin{bmatrix} 0 & .5 & 2.105 & 6 & 8 & 7.5 & 5.895 & 2 \\ 0 & 0 & 6.42 & 0 & 8 & 8 & 1.58 & 8 \end{bmatrix}$$



Homogeous Coordinates :

changing from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$

if we have point $(x,y) \in \mathbb{R}^2$ we can identify it with point $(x,y,1) \in \mathbb{R}^2$

any # as long as constant

ex : $(1,-3)$ has homogeneous coordinates

$(1,-3,1)$

#s not changed – we just have extra term at the end now

* Get these coordinates by transformation.

ex: $\mathbb{R}^2$ transformation $(x,y) \longmapsto (x+h, y+k)$ can be written in homogeneous coordinates as $(x,y,1) \longmapsto (x+h, y+k, 1)$

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+h \\ y+k \\ 1 \end{bmatrix}$$

In general:

any $\mathbb{R}^2$ transformation given by $T(\bar{x}) = A\bar{x}$ can be written in homogeneous coordinates by block matrix of the form $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$

into $\mathbb{R}^3$

$$\underbrace{\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\circlearrowleft \text{ angle} = \varphi}, \quad \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{reflection over } y = x}, \quad \underbrace{\begin{bmatrix} s & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{scale } x \text{ by } s \\ \text{scale } y \text{ by } t}}$$

Composite Transformations

using more than 1 basic transformation

ex: find a 3×3 matrix that coorresponds to the composite transformation of a scaling of 4, a rotation of 90°, and a translation which adds (2, -1) to each point on the figure.

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{scaling}} \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{rotation} \\ \varphi = 90^\circ}} \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{translation}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & -4 & 4 \\ 4 & 0 & 8 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \text{multiplied}$$