

3.1: Intro to Determinants

2x2 determinant:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det A = ad - bc$$

3x3 determinant:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} \nearrow - \nearrow - \nearrow - \\ \searrow + \searrow + \searrow + \end{matrix}$$

$$= (a_{11}a_{22}a_{33}) + (a_{12}a_{23}a_{31}) + (a_{13}a_{21}a_{32}) \\ - (a_{31}a_{22}a_{13}) - (a_{32}a_{23}a_{11}) - (a_{33}a_{21}a_{12})$$

ex: find

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix} \begin{matrix} \nearrow 1 \nearrow 3 \\ \searrow 2 \searrow 1 \\ \searrow 3 \searrow 4 \end{matrix}$$

$$= (2 + 9 + 40) - (15 + 4 + 12) \\ = 51 - 31 \\ = 20$$

Def: $\det A$, $n \geq 2$, A is $n \times n$

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \dots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j}$$

expanded
along 1st row

example: find $\det A$

$$A = \begin{bmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{bmatrix}$$

Sign matrix: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$$\det A = 3 \cdot \det A_{11} - 0 \cdot \det A_{12} + 4 \cdot \det A_{13}$$

$$\det A_{11} = \begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix} \quad \begin{bmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{bmatrix}$$
$$= -3 - 10 = -13$$

$$\det A_{12} = \begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} \quad \begin{bmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{bmatrix}$$
$$= -2 - 0 = -2$$

$$\det A_{13} = \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} \quad \begin{bmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{bmatrix}$$
$$= 10 - 0 = 10$$

$$\text{So } \det A = 3(-13) + 4(10) = -39 + 40 = 1$$

Cofactor: $C_{ij} = (-1)^{i+j} \det A_{ij}$

Thrm: Cofactor Expansion

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in} \quad \leftarrow \text{expand on } i^{\text{th}} \text{ row}$$

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj} \quad \leftarrow \text{expand on } j^{\text{th}} \text{ Column}$$

ex: Compute

$$\begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix}$$

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$= +2 \cdot \begin{vmatrix} 0 & 0 & \textcircled{5} \\ 7 & 2 & -5 \\ 3 & 1 & 8 \end{vmatrix}$$

$$= 2 \cdot 5 \cdot \begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= 10 (7 - 6) = 10$$

We can expand on any row or column - smart: find row/column w/ most 0
* we want to expand along 3rd row

ex: Compute

$$\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$$

$$\begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{bmatrix}$$

$$= -2 \cdot \begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix} = -2 \cdot 3 \cdot \begin{vmatrix} 4 & 3 & -5 & 4 & 3 \\ 5 & 2 & -3 & 5 & 2 \\ 0 & -1 & 2 & 0 & -1 \end{vmatrix}$$

$$= -6 \cdot [(16 + 0 + 25) - (0 + 12 + 30)]$$

$$= -6 (41 - 42) = 6$$

Thm: if A is triangular, then $\det A$ is the product of the entries on the main diagonal

ex: compute

$$\begin{vmatrix} 2 & 3 & -4 & 0 & -1 \\ 0 & -5 & 2 & -3 & 6 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 8 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix} \begin{bmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{bmatrix}$$

$$= 3 \cdot \begin{vmatrix} 2 & 3 & -4 & 0 \\ 0 & -5 & 2 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 8 \end{vmatrix} = 3 \cdot 8 \cdot \begin{vmatrix} 2 & 3 & -4 \\ 0 & -5 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 24 \cdot 1 \cdot \begin{vmatrix} 2 & 3 \\ 0 & -5 \end{vmatrix} = 24 (-10 - 0)$$

$$= -240$$

$$\& \text{ product of diagonal} = 2 \cdot -5 \cdot 1 \cdot 8 \cdot 3 = -240$$