3.2: Properties of Determinants

Thm 3: Row Operations

Let A be $n \times n$ matrix

1) if multiple of 1 row of A is added to another row of A to form B, then
$$\det B = \det A$$

2) if 2 rows of A are switched to form B then
$$\det B = -\det A$$

3) if one row of A is multiplied by k to form B, then
$$\det B = k \cdot \det A$$

Using the definition of $\det A$ is inefficient for Large matrices.

Strategy: reduce A to echelon form
find determinent of the Δ matrix

ex: compute

$$\begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{vmatrix} \begin{array}{l} \\ \\ R_3 \leftrightarrow R_3 + 2R_1 \\ R_4 \leftrightarrow R_4 + 3R_1 \end{array} = \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ 0 & 2 & 4 & -10 \end{vmatrix} \begin{array}{l} \\ \\ R_2 + R_3 \\ R_4 - 2R_2 \end{array}$$

$$= \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

having a row or column of zeros $\Rightarrow \det A = 0$

ex: Compute

$$\begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} \begin{array}{l} \\ R_2-3R_1 \\ R_3+R_2 \\ R_4-R_1 \end{array}$$

$$= 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -9 & 6 & 8 \\ 0 & 0 & -3 & 2 \end{vmatrix} \begin{array}{l} \\ \\ R_3+3R_2 \\ \\ \end{array} = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

$$= 2 \cdot -2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -3 & 2 \end{vmatrix} \begin{array}{l} \\ \\ \\ R_4+R_3 \end{array} = -4 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= -4 \cdot (1 \cdot 3 \cdot 3 \cdot 1) = -4 \cdot 9 = -36$$

$$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix}, & A \text{ invertible} \\ 0, & A \text{ singular} \end{cases}$$

where U is echelon form of A

A invertible $\Leftrightarrow$ $\det A \neq 0$

$\det A = \det A^T$

- A & B square $n \times n$ matrices then $\det AB = \det A \det B$

ex:

$$A = \begin{bmatrix} 4 & -3 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -3 \\ -2 & 10 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & -3 \\ -2 & 10 \end{bmatrix} = \begin{bmatrix} 6 & -42 \\ 4 & -23 \end{bmatrix}$$

$$\det(AB) = (6 \cdot -23 - 4 \cdot -42) = -138 + 168 = 30$$

$$\det A \cdot \det B = (-8 - (-3))(0 - 6) = (-5)(-6) = 30$$

Warning: $\det(A+B) \neq \det A + \det B$