Cramer's Rule

Solve $A\bar{x} = \bar{b}$, A is $n \times n$

define: $A_i(\bar{b}) = [\bar{a}_1 \ \bar{a}_2 \ \cdots \ \bar{b} \ \cdots \ \bar{a}_n]$

i^{th} column

replace $\bar{a}_i$ with $\bar{b}$

Rule:

Let A be an $n \times n$ invertible matrix. For any $\bar{b} \in \mathbb{R}^n$, the unique solution $\bar{x}$ of $A\bar{x} = \bar{b}$ can be found by:

$$x_i = \frac{\det A_i(\bar{b})}{\det A} \quad ; \quad i = 1, 2, \ldots, n$$

note: Cramer's Rule is only efficient for small matrices, typically only for 3x3

ex: Use Cramer's Rule to solve

$$\begin{cases} 4x_1 + x_2 = 6 \\ 5x_1 + 2x_2 = 7 \end{cases} \Longrightarrow A = \begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}, \ b = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 6 & 1 \\ 7 & 2 \end{vmatrix}}{\begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix}} = \frac{12 - 7}{8 - 5} = \frac{5}{3}$$

$$x_2 = \frac{\begin{vmatrix} 4 & 6 \\ 5 & 7 \end{vmatrix}}{\begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix}} = \frac{28 - 30}{3} = \frac{-2}{3}$$

So $\bar{x} = \begin{bmatrix} 5/3 \\ -2/3 \end{bmatrix}$

ex: Solve using Cramer's Rule, for which s is solution unique?

$$\begin{cases} 6s x_1 + 4x_2 = 5 \\ 9x_1 + 2s x_2 = -2 \end{cases}$$

$$x_1 = \frac{\begin{vmatrix} 5 & 4 \\ -2 & 2s \end{vmatrix}}{\begin{vmatrix} 6s & 4 \\ 9 & 2s \end{vmatrix}} = \frac{10s+8}{12s^2-36} = \frac{\cancel{2}(5s+4)}{12(s^2-3)} = \frac{5s+4}{6(s-\sqrt{3})(s+\sqrt{3})}$$

$$x_2 = \frac{\begin{vmatrix} 6s & 5 \\ 9 & -2 \end{vmatrix}}{\begin{vmatrix} 6s & 4 \\ 9 & 2s \end{vmatrix}} = \frac{-12s-45}{12(s-\sqrt{3})(s+\sqrt{3})} = \frac{-4s-15}{4(s-\sqrt{3})(s+\sqrt{3})}$$

So $s \neq \pm\sqrt{3}$

and

$$\bar{x} = \begin{bmatrix} \frac{5s+4}{6(s^2-3)} \\ \frac{-4s-15}{4(s^2-3)} \end{bmatrix}$$

Formula for A^{-1}:

def: Adjugate (or adjoint) Matrix

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & & \vdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix}^T$$ matrix of cofactors

$$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj}(A)$$

ex: find A^{-1} if $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

find cofactors:

$C_{11} = +\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 5$ $\qquad C_{12} = -\begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} = 2$

$C_{13} = +\begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} = -4$ $\qquad C_{21} = -\begin{vmatrix} 6 & 7 \\ 3 & 4 \end{vmatrix} = -3$

$C_{22} = +\begin{vmatrix} 3 & 7 \\ 2 & 4 \end{vmatrix} = -2$ $\qquad C_{23} = -\begin{vmatrix} 3 & 6 \\ 2 & 3 \end{vmatrix} = 3$

$C_{31} = +\begin{vmatrix} 6 & 7 \\ 2 & 1 \end{vmatrix} = -8$ $\qquad C_{32} = -\begin{vmatrix} 3 & 7 \\ 0 & 1 \end{vmatrix} = -3$

$C_{33} = +\begin{vmatrix} 3 & 6 \\ 0 & 2 \end{vmatrix} = 6$

$$\det A = 3\,C_{11} + 6\,C_{12} + 7\,C_{13} = 3(5) + 6(2) + 7(-4)$$
$$= 15 + 12 - 28 = -1$$

$$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A = -1 \cdot \begin{bmatrix} 5 & 2 & -4 \\ -3 & -2 & 3 \\ -8 & -3 & 6 \end{bmatrix}^T$$

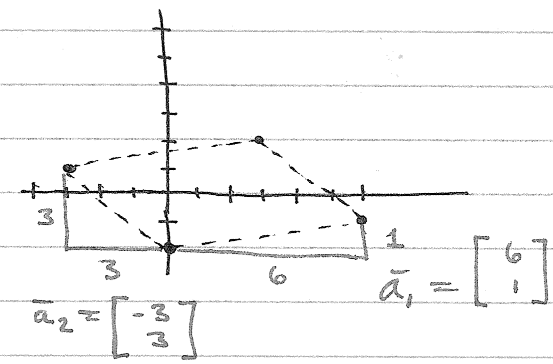
$$= \begin{bmatrix} -5 & -2 & 4 \\ 3 & 2 & -3 \\ 8 & 3 & -6 \end{bmatrix}^T = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}$$

Volume:

$|\det A|$ = volume of the parallelpiped determined by the columns of A

in 2D

find the area of the parallogram determined by the points: $(0,-2), (6,-1), (-3,1), (3,2)$



$$A = \begin{bmatrix} 6 & -3 \\ 1 & 3 \end{bmatrix}$$

$$\text{area} = |\det A| = |18 - (-3)| = 21$$

Volume: $T(S)$

- $T: \mathbb{R}^2 \to \mathbb{R}^2$ Linear transformation & S parallelogram then $\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$

- $T: \mathbb{R}^3 \to \mathbb{R}^3$ linear transformation & S parallelpiped then $\{$~~area~~ Volume of $T(S)\} = |\det A| \cdot \{$volume of $S\}$