4.1 : vector spaces + subspaces

Vector Space

a non-empty set V of objects called vectors on which there are 2 opperations: addition & multiplication by scalers, which must hold to the axioms below. These rules must hold (or be true) for all vectors $\bar{u}, \bar{v}, \bar{w}$ in V and for all scalers c & d.

1. the sum of $\bar{u}$ & $\bar{v}$ is $\bar{u}+\bar{v} \in V$
   (ie: $\forall\, \bar{u}, \bar{v} \in V \Rightarrow \bar{u}+\bar{v} \in V$)
2. $\bar{u}+\bar{v} = \bar{v}+\bar{u}$
3. $(\bar{u}+\bar{v})+\bar{w} = \bar{u}+(\bar{v}+\bar{w})$
4. there is a zero vector $\bar{0}$ in V such that
   $\bar{u}+\bar{0} = \bar{u}$ (ie: $\exists\, \bar{0} \in V \ni \bar{u}+\bar{0} = \bar{u}$)
5. $\forall\, \bar{u} \in V$, $\exists$ a vector $-\bar{u} \in V \ni \bar{u}+(-\bar{u}) = \bar{0}$
6. scaler multiple: $c \cdot \bar{u}$ is in V
7. $c(\bar{u}+\bar{v}) = c\bar{u}+c\bar{v}$
8. $(c+d)\bar{u} = c\bar{u}+d\bar{u}$
9. $c(d\bar{u}) = (cd)\bar{u}$
10. $1 \cdot \bar{u} = \bar{u}$

Results:

$\bar{0}$ is unique

1 is unique

for each $\bar{u} \in V$, $-\bar{u}$ is unique

And

$0 \cdot \bar{u} = \bar{0}$

$c \cdot \bar{0} = \bar{0}$

$-\bar{u} = (-1)\bar{u}$

examples

1. $\mathbb{R}^n$, $n \geq 1$

2. V = set of all arrows in 3D spaces where 2 arrows are considered equal if they have the same length & point in the same direction

3. $\mathbb{P}_n$, $n \geq 0$
   the set of polynomials of degree at most n
   $$p(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$$
   note: $\deg p(t) = 0$ if $p(t) = a_0 \neq 0$

4. V = set of all real-valued functions

Subspaces

a subset H of a vector space V is a subspace if

(1) the zero vector of V is in H

(2) H is closed under vector addition
(ie: $\bar{u}, \bar{v} \in H \Rightarrow \bar{u} + \bar{v} \in H$)

(3) H is closed under scaler multiplication
(ie: $\bar{u} \in H \Rightarrow c \cdot \bar{u} \in H$, $\forall c$)

Note: because H is inside of V (or contained in V) and if all 3 of the above properties hold then H is also a vector space.

ex: $H = \{\bar{0}\}$ in Vector space V

H is called the zero subspace

why?

(1) $\bar{0} \in H$ ✓

(2) take 2 vectors in H & add them:

$$\bar{0} + \bar{0} = \bar{0} \in H \checkmark$$

(3) c is any scaler & $\bar{0} \in H$

$$\Rightarrow c \cdot \bar{0} = \bar{0} \in H$$

So H is a subspace of V

ex: $\mathbb{P}$ = set of all polynomials w/ $\mathbb{R}$ coeff.

w/ operations defined on functions

$\mathbb{P}$ is a subset of the set of all real valued functions

$p(t) = 0$ is in $\mathbb{P}$

$p(t) + q(t)$ is a polynomial (p(t) arbitrary poly in $\mathbb{P}$)

& $c \cdot p(t)$ is a polynomial

$\therefore$ $\mathbb{P}$ is a subspace of the vector space of real-valued functs.

(& a vector space on its own)

ex: $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$

ex: $H = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$

why?

1) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H$, $a=b=0$

2) $\bar{u} + \bar{v} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ d \\ 0 \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \\ 0 \end{bmatrix} \in H$

3) $c \cdot \bar{u} = c \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} ac \\ bc \\ 0 \end{bmatrix} \in H$

ex: a line not thru the origin in $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^2$

ex: a plane in $\mathbb{R}^3$ not thru the origin is not a subspace of $\mathbb{R}^3$

ex: given vectors v_1 & v_2 in V
define $H = \text{span}\{v_1, v_2\}$
is H a subspace of V?

① $0 = 0 \cdot v_1 + 0 \cdot v_2 \in H$

② $\bar{u} = a_1 v_1 + a_2 v_2$, $\bar{w} = b_1 v_1 + b_2 v_2$

$\bar{u} + \bar{w} = a_1 v_1 + a_2 v_2 + b_1 v_1 + b_2 v_2$

$= (a_1 + b_1) v_1 + (a_2 + b_2) v_2 \in H$

③ $c \cdot \bar{u} = c(a_1 v_1 + a_2 v_2) = (ca_1) v_1 + (ca_2) v_2 \in H$

Answer: yes

Thrm: if $v_1, \dots, v_p \in V$ a vector space
then $\text{Span}\{v_1, \dots, v_p\}$ is a subspace of V

example:
Let $H = \{(2s+t,\ s-4t,\ 3s+2t,\ t) \mid s,t \in \mathbb{R}\}$
Show H is a subspace of $\mathbb{R}^4$

an arbitary vector in H (in Column form):

$$\begin{bmatrix} 2s+t \\ s-4t \\ 3s+2t \\ t \end{bmatrix} = \begin{bmatrix} 2s \\ s \\ 3s \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ -4t \\ 2t \\ t \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -4 \\ 2 \\ 1 \end{bmatrix}$$

def: $\bar{u} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}$ & $\bar{v} = \begin{bmatrix} 1 \\ -4 \\ 2 \\ 1 \end{bmatrix}$

So, $H = \text{span}\{\bar{u}, \bar{v}\} = \{s \cdot \bar{u} + t \cdot \bar{v} \mid s,t \in \mathbb{R}\}$

now, H is a subset of $\mathbb{R}^4$
(ie: length 4 vectors)

Conclusion: H is a subspace of $\mathbb{R}^4$

example:
Let $K = \{(2a+b,\ b-1) \mid a,b \in \mathbb{R}\}$
Is K a subspace of $\mathbb{R}^2$?