Null Space

$$\text{Nul } A = \{ \bar{x} \mid \bar{x} \in \mathbb{R}^n, \ A\bar{x} = \bar{0} \}$$

the Null space of an $m \times n$ matrix A is the set of all solutions of $A\bar{x} = \bar{0}$

ex:

$$\begin{cases} 2x_1 + x_2 - 3x_3 = 0 \\ x_1 - 2x_2 + 5x_3 = 0 \end{cases}$$

$$\Rightarrow A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 5 \end{bmatrix}$$

Let $\bar{u} = \begin{bmatrix} 1 \\ 13 \\ 5 \end{bmatrix}$

Is $\bar{u}$ in the Null space of A?

Solution: check if $A\bar{u} = \bar{0}$

$$A\bar{u} = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 13 \\ 5 \end{bmatrix} = \begin{bmatrix} 2+13-15 \\ 1-26+25 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \bar{u} \in \text{Nul}(A)$$

Thrm:
The Nul space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

Describing Nul(A)

Steps:

① start with a matrix A

$$A = \begin{bmatrix} 2 & -4 & 5 & 8 & -4 \\ -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \end{bmatrix}$$

② row reduce the matrix $[A \mid \bar{0}]$

$$[A \mid \bar{0}] \rightsquigarrow \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

③ Write the solution vector of $A\bar{x} = \bar{0}$

$$\begin{cases} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \end{cases}$$

$$\Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 + 3x_5 \\ + x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= \underset{\substack{\uparrow \\ u}}{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} x_2 + \underset{\substack{\uparrow \\ v}}{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}} x_4 + \underset{\substack{\uparrow \\ w}}{\begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}} x_5$$

$Nul(A) = span(u, v, w)$

Column Space of A

the column space of an $m \times n$ matrix A is the set of all linear combinations of the columns of A:

$$\text{Col}(A) = \text{Span}\{\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n\}$$

Thrm: Col(A) is a subspace of $\mathbb{R}^m$

Note: $\text{Col}(A) = \{b \mid b = A\bar{x},\ x \in \mathbb{R}^n\}$
so Col(A) is the range of the linear transformation $T(x) = Ax$

ex: find a matrix A such that $W = \text{Col}(A)$

$$W = \left\{ \begin{bmatrix} a-4b \\ 3a+b \\ b-7a \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

Solution:

$$W = \left\{ \begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix} a + \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} b : a, b \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \right\}$$

So, we can pick

$$A = \begin{bmatrix} 1 & -4 \\ 3 & 1 \\ -7 & 1 \end{bmatrix}$$

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<u>example</u>:

$$A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 0 & -5 \\ 0 & -5 & 7 \\ -5 & 7 & -2 \end{bmatrix}$$

a) If the column space of A is a subspace of $\mathbb{R}^k$, what is k?

$k = 4$, since columns of A are in $\mathbb{R}^4$

b) If $Nul(A)$ is a subspace of $\mathbb{R}^k$, what is k?

$k = 3$, since $x \in Nul(A)$ if $A\bar{x} = 0$

c) Find a non-zero vector in $Col(A)$

$\begin{bmatrix} 7 \\ -2 \\ 0 \\ -5 \end{bmatrix}$, ie: pick any column of A

d) Find a non-zero vector in $Nul(A)$

$$[A \mid \bar{0}] \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

impossible since $Nul(A) = \{\bar{0}\}$

e) is $u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ in $Nul(A)$?

$A\bar{u} = \begin{bmatrix} 9 \\ -2 \\ 5 \\ -12 \end{bmatrix} \neq \bar{0}$, No

f) is $v = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ in Col(A)?

$$[A \mid v] = \begin{bmatrix} 7 & -2 & 0 & 2 \\ -2 & 0 & -5 & -3 \\ 0 & -5 & 7 & 0 \\ -5 & 7 & -2 & 1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 92/203 \\ 0 & 1 & 0 & 17/29 \\ 0 & 0 & 1 & 85/203 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ yes, it's consistent so we have a solution

means: $v \in \text{Col}(A)$

Range & Kernal of Linear Combination

Linear Transformation:

$T: V \to W$, V & W vector spaces

with

$T(u+v) = T(u) + T(v)$

& $T(cu) = cT(u)$

Kernal of T (Null space) is the set of all $u \in V$ such that $T(u) = 0$

Range of T (column space) is the set of all $T(u)$ such that $u \in V$