4.3: Linearly Independent Sets, bases

idea: Looking at sets which span V (vector space) the most "effeciently"

Linearly Independent

given ~~get~~ a set $\{\bar{v}_1, \dots, \bar{v}_p\}$ in V
find constants $c_1, c_2, \dots, c_p$ such that

$$c_1 \bar{v}_1 + c_2 \bar{v}_2 + \dots + c_p \cdot \bar{v}_p = 0$$

if $c_1 = c_2 = \dots = c_p = 0$ is the only solution, then $\{\bar{v}_1, \dots, \bar{v}_p\}$ are linearly Independent

else they're linearly dependent

Thm:

$\{v_1, \dots, v_p\}$, a set of 2+ vectors, is Linearly dependent iff for some $j > 1$, v_j is a Linear combination of $v_1, v_2, \dots, v_{j-1}$ (assumption: $v_1 \neq 0$)

* Note: in a general vector space we won't always be able to put the "vectors" into a matrix & row reduce

ex: $\bar{p}_1(t) = 1$, $\bar{p}_2(t) = 2 - t$, & $\bar{p}_3(t) = 2 + t$
Is $\{\bar{p}_1, \bar{p}_2, \bar{p}_3\}$ linearly dependent or indep?

b/c

$$p_3 = 4p_1 - p_2$$

ex: $\{\sin t, \cos t\}$ is Linearly independent in $C[0,1]$

where $C[0,1] = \left\{ f \;\middle|\; \begin{array}{l} f \text{ is a continuous function} \\ \text{on } [0,1] \end{array} \right\}$

if dependent, we would need:

$$\sin t = \underbrace{c}_{\text{just a constant}} \cdot \cos t$$

no such c exists

Def: Basis

H is a subspace of ~~subve~~ some vector space, V,
Let $\mathcal{B} = \{b_1, b_2, \ldots, b_p\}$ is a set of indexed vectors in V.
$\mathcal{B}$ is a basis of/for H if

(1) $\mathcal{B}$ is Linearly Independent

(2) $H = \operatorname{span}\{b_1, b_2, \ldots, b_p\} = \operatorname{span} \mathcal{B}$

(and obviously $\mathcal{B} \subseteq H$)

ex: the standard basis for $\mathbb{R}^n$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

ex: $V_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $V_3 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$

Show $\{V_1, V_2, V_3\}$ is a basis for $\mathbb{R}^3$

1 method:

create: $\begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 3 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

so we have a pivot in each row

& a pivot in each column

$\Rightarrow$ Independent & Spans $\mathbb{R}^3$

ex: Standard Basis for $\mathbb{P}_n$ ← polynomials of degree n or less

$S = \{1, t, t^2, \ldots, t^n\}$

obviously, S spans $\mathbb{P}_n$

Independent ?

$$c_1 \cdot 1 + c_2 \cdot t + c_2 \cdot t^2 + \ldots + c_n \cdot t^n = 0$$

the zero polynomial

compare like terms

so $c_1 = c_2 = \ldots = c_n = 0$

$\therefore$ S is a basis for $\mathbb{P}_n$

Spanning Set Thrm:

ex: $V_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$, $V_2 = \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$, $V_3 = \begin{bmatrix} 5 \\ -10 \\ -5 \end{bmatrix}$, $H = \text{Span}\{V_1, V_2, V_3\}$

Show $\text{Span}\{V_1, V_2, V_3\} = \text{Span}\{V_1, V_2\}$
& find ~~a~~ a basis for H

note: $3V_1 + 2V_2 = V_3$

step 1: every vector in $\text{Span}\{V_1, V_2\}$ is in H

$$\underbrace{C_1V_1 + C_2V_2}_{\text{in span}\{V_1,V_2\}} = \underbrace{C_1V_1 + C_2V_2 + 0 \cdot V_3}_{\text{in } H}$$

step 2: every vector in H, call it X is in $\text{Span}\{V_1, V_2\}$

$$\overset{\text{in } H}{X} = C_1V_1 + C_2V_2 + C_3V_3$$
$$= C_1V_1 + C_2V_2 + C_3(3V_1 + 2V_2)$$
$$= C_1V_1 + C_2V_2 + 3C_3V_1 + 2C_3V_2$$
$$= \underbrace{(C_1 + 3C_3)V_1 + (C_2 + 2C_3)V_2}_{\text{in Span}\{V_1, V_2\}}$$

So $\text{Span}\{V_1, V_2, V_3\} = \text{Span}\{V_1, V_2\}$

OR $H = \text{Span}\{V_1, V_2\}$

guess: $\{V_1, V_2\}$ basis of H

1. $\text{Span}\{V_1, V_2\} = H$ ✓
2. $\{V_1, V_2\}$ independent?

yes, b/c not multiples

Spanning Set Thm:

Let $S = \{v_1, v_2, \dots, v_p\}$ be a set in V,
and let $H = \text{Span}\{v_1, \dots, v_p\}$

(a) If one of the vectors in S is a linear combination of the other vectors, then if you remove that vector from S, the new set still spans H

(b) If $H \neq \{0\}$, some subset of S is a basis for H

* means:

you find a set which spans your vector space.
Then keep removing vectors until you get a linearly independent set.
That set is your basis.

Bases for $Nul(A)$

write the solution of $A\bar{x} = \bar{0}$ in parametric vector form

the vectors of the coefficients of the free variables are your basis vectors

Bases for Col(A)

- pivot columns of A form a basis for $\text{Col}(A)$

Note: row reductions don't affect Linear independence / dependence

ex: find a basis for

$$B = \begin{array}{c} \begin{bmatrix} 1 & 0 & -5 & 0 & 7 \\ 0 & 1 & -4 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & & \uparrow & \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{array} \end{array}$$

non pivot columns:

$$\begin{cases} b_3 = -5b_1 - 4b_2 \\ b_5 = 7b_1 + 6b_2 - 3b_4 \end{cases}$$

$$\text{basis of Col}(B) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

ex: find a basis for

$$A = \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & & \uparrow & \end{array} \end{array}$$

pivot columns

where $\text{rref}(A) = B$

Note:

$$a_3 = -5a_1 - 4a_2$$

$$a_5 = 7a_1 + 6a_2 - 3a_4$$

$$\text{basis for Col}(A) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Notice:

$$\left\{ \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -5 \end{bmatrix} \right\}, \quad \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Lin Independent — basis of $\mathbb{R}^3$ — spans $\mathbb{R}^3$