The Unique Representation Theorem:

Let $\mathcal{B} = \{\bar{b}_1, \dots, \bar{b}_n\}$ be a basis for vector space V

Then $\forall \bar{x} \in V$, $\exists!$ set of scalers $c_1, \dots, c_n$ $\ni$

$$\bar{x} = c_1 \bar{b}_1 + c_2 \bar{b}_2 + \dots + c_n \bar{b}_n$$

pf:

since $\mathcal{B}$ is a basis we know scalers exist

so suppose they're not unique:

$$\bar{x} = d_1 \bar{b}_1 + d_2 \bar{b}_2 + \dots + d_n \bar{b}_n$$

and

$$\bar{x} = c_1 \bar{b}_1 + c_2 \bar{b}_2 + \dots + \bar{c}_n \bar{b}_n$$

then

$$\bar{0} = (\bar{x} - \bar{x}) = (d_1 \bar{b}_1 + \dots + d_n \bar{b}_n) - (c_1 \bar{b}_1 + \dots + c_n \bar{b}_n)$$

$$= (d_1 - c_1)\bar{b}_1 + \dots + (d_n - c_n)\bar{b}_n$$

but $\mathcal{B}$ is a basis

$$\Rightarrow \quad (d_i - c_i) = 0 \quad \forall i$$

$$\Rightarrow \quad d_i = c_i \quad \forall i$$

Def: $\mathcal{B}$ - coordinates of $\bar{x}$

$\mathcal{B} = \{b_1, \dots, b_n\}$ is a basis of V and $x \in V$

The coordinates of x relative to the basis $\mathcal{B}$ (ie: the $\mathcal{B}$-coordinates of $\bar{x}$) are the coefficients $c_1, c_2, \dots, c_n$ $\ni$ $\bar{x} = c_1 \bar{b}_1 + c_2 \bar{b}_2 + \dots + c_n \bar{b}_n$

notation:

$$\underbrace{[x]_{\mathcal{B}}}_{\text{x in basis } \mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

ex: $\mathcal{B} = \{ b_1, b_2 \}$ for $\mathbb{R}^2$

$b_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $b_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $[x]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

find $\bar{x}$ in the std basis of $\mathbb{R}^2$

$$\bar{x} = c_1 \bar{b}_1 + c_2 \bar{b}_2 = 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Note: std basis

$$\mathcal{E} = \{ e_1, e_2 \}$$

$$[x]_{\mathcal{E}} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-2)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So

$$x = [x]_{\mathcal{E}}$$

in practice, we drop the $\mathcal{E}$

What does this change of basis mean?

(illustration → p 247 – 248)

it's just a different way to write the vector
→ often it's easier to use in a specific basis

Coordinates in $\mathbb{R}^n$

finding $[x]_{\mathcal{B}}$ where $\mathcal{B}$ is a basis of $\mathbb{R}^n$

<u>ex</u>: $\mathcal{B} = \{b_1, b_2\}$ basis of $\mathbb{R}^2$

$$b_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \text{find } [x]_{\mathcal{B}}$$

solution:

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \text{such that}$$

$$x = c_1 b_1 + c_2 b_2$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} c_2$$

$$\Rightarrow \begin{array}{c} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \\ \begin{matrix} b_1 & b_2 \end{matrix} \end{array} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad c_2 = -2 \quad \& \quad c_1 = 5$$

$$\text{So,} \quad [x]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Change of Coordinates Matrix

$$P_{\mathcal{B}} = [\, b_1 \;\; b_2 \;\; b_3 \;\; \dots \;\; b_n \,]$$

<u>changes</u> $\mathcal{B}$-coordinates $\Rightarrow$ std basis

$$\boxed{x = P_{\mathcal{B}} \cdot [x]_{\mathcal{B}}}$$

note $P_{\mathcal{B}}$ is always invertible

$$\text{So} \quad \boxed{[x]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} x}$$

Coordinate Mapping

Basis $\mathcal{B} = \{b_1, \dots, b_n\}$ for vector space V
puts a coordinate system onto V (ie: relative to $\mathcal{B}$)

the map $x \mapsto [x]_{\mathcal{B}}$ is called the coordinate map
& it goes from $V \longrightarrow \mathbb{R}^n$

<u>Thrm</u>

Let $\mathcal{B} = \{b_1, \dots, b_n\}$ be a basis for a vector space V. Then the coordinate map $x \mapsto [x]_{\mathcal{B}}$ is a 1-1 linear transformation from V onto $\mathbb{R}^n$

<u>ex</u>: Vector Space $= \mathbb{P}_3 = \{ a_0 + a_1 t + a_2 t^2 + a_3 t^3 \mid a_i \in \mathbb{R} \}$
& basis $\mathcal{B} = \{1, t, t^2, t^3\}$

if $p \in \mathbb{P}_3 \implies p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

then $[p]_{\mathcal{B}} = \begin{array}{cc} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} & \begin{array}{l} \leftarrow 1 \\ \leftarrow t \\ \leftarrow t^2 \\ \leftarrow t^3 \end{array} \end{array}$

here the coordinate mapping $p \mapsto [p]_{\mathcal{B}}$
is 1-1 and a linear transformation
from $\mathbb{P}_3$ onto $\mathbb{R}^4$

* we can think of the coordinate mapping as just a different way to write the original object

ex: use coordinate vectors to verify that the polynomials $(1+t^3)$, $(3+t-2t^2)$, and $(-t+3t^2-t^3)$ are linearly independent in $\mathbb{P}_3$

$$(1+t^3) \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad (-t+3t^2-t^3) \Rightarrow \begin{bmatrix} 0 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

$$(3+t-2t^2) \Rightarrow \begin{bmatrix} 3 \\ 1 \\ -2 \\ 0 \end{bmatrix}$$

Solve for x in $Ax = 0$:

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 3 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

ex: $v_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 5 \\ -9 \end{bmatrix}$

$H = \text{span}\{v_1, v_2\}$ and $\{v_1, v_2\} = \mathcal{B}$ is a basis of H

if $x \in H$, find $[x]_{\mathcal{B}}$

Solution:

$$c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -9 \end{bmatrix}, \text{ where } [x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & -1 & 5 \\ 0 & 3 & -9 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} c_1 = 2 \\ c_2 = -3 \end{array}$$

So $[x]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$