4.5: dimension of a vector space

Thrm:

if a vector space V has a basis $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$, then any set in V which has more than n vectors must be linearly dependent.

why?

step 1: ~~put/write~~ we have a set $\{u_1, \dots, u_p\}$, $p > n$
it has more than n vectors
write each u_i in terms of $\mathcal{B}$
ie: $\{[u_1]_\mathcal{B}, [u_2]_\mathcal{B}, \dots, [u_p]_\mathcal{B}\}$

step 2: for each $[u_i]_\mathcal{B}$ this new vector has length* n
(b/c n vectors in $\mathcal{B}$)
but there are $p > n$ of them
$\Rightarrow \{[u_1]_\mathcal{B}, \dots, [u_p]_\mathcal{B}\}$ is linearly dependent

step 3: we can find $c_1, \dots, c_p$ not all zero such that

$$c_1 [u_1]_\mathcal{B} + \dots + c_p [u_p]_\mathcal{B} = \bar{0}$$

step 4: the coordinate mapping is a linear transformation:

$$[c_1 u_1 + \dots + c_p u_p]_\mathcal{B} = \bar{0}$$

$\Rightarrow c_1 u_1 + \dots + c_p u_p = 0 \cdot b_1 + \dots + 0 \cdot b_n = \bar{0}$

$\Rightarrow \{u_1, \dots, u_p\}$ is linearly dependent

Thrm

If a vector space V has a basis of n vectors, then every basis of V also has n vectors.

Why?

- if a set has more than n vectors it must be dependent (see previous thrm)
- if a set has less than n vectors then it can't span V, since you wouldn't be able to make one of the vectors in the original basis (think linear indepence)

Def finite- & infinite-dimensional

- If V is spanned by a finite set, then V is finite-dimensional & dim V = the number of vectors in V's basis
- $\dim\{0\} = 0$
- If V is not spanned by a finite set, it is infinite-dimensional

examples:

$\mathbb{R}^n$ the std basis has n vectors $\Rightarrow \dim \mathbb{R}^n = n$

$\mathbb{C}$ a std basis is $\{1, i\} \Rightarrow \dim \mathbb{C} = 2$

$\mathbb{P}_2$ the std basis is $\{1, t, t^2\} \Rightarrow \dim \mathbb{P}_2 = 3$

$\mathbb{P}_n$ the std basis is $\{1, t, \dots, t^n\} \Rightarrow \dim \mathbb{P}_n = n+1$

$\mathbb{P}$ this is the set of <u>all</u> polys $\Rightarrow \dim \mathbb{P} = \infty$

ex: Find the dimension of H (a subspace of $\mathbb{R}^4$)

$$H = \left\{ \begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}$$

H is the set of all linear combinations of

$$\underset{\text{"a"}}{v_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}}, \quad \underset{\text{"b"}}{v_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}}, \quad \underset{\text{"c"}}{v_3 = \cancel{\begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}}}, \quad \underset{\text{"d"}}{v_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}}$$

obviously, $H = \text{span}\{v_1, v_2, v_3, v_4\}$

Now, if there are any ~~depen~~ vectors which are linear combinations of the others, we remove them:

v_4 – keep b/c of the 5

v_3 – note $v_3 = -2v_2 \Rightarrow$ remove

v_2 & v_1 – not multiples of each other $\Rightarrow$ keep them

$\therefore$ $\{v_1, v_2, v_4\}$ is now a linearly independent set; thus, it's also the basis of H

$$\Rightarrow \dim H = 3$$

ex: Subspaces of $\mathbb{R}^3$

0-dim subspace: Only the zero subspace

1-dim subspaces: any straight line thru $(0,0,0)$

2-dim subspaces: any plane thru the origin

3-dim subspace: Only $\mathbb{R}^3$ itself

Thrm

H is a subspace of a finite-dim vector space V.
(any linearly independent set in H can be expanded to a basis for H)
$\Rightarrow$ H is finite-dim and

$$\dim H \le \dim V$$

* Thrm

let V be a p-dim vector space $(p \ge 1)$
$\Rightarrow$ any linearly independent set of exactly p elements in V is automatically a basis
& any set of exactly p elements that spans V is automatically a basis for V

Dimension of Nul(A)

$\dim(Nul(A))$ = # of free variables in $A\bar{x} = \bar{0}$

Dimension of Col(A)

$\dim Col(A)$ = # of pivots in A

ex:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

$$[A|0] = \left[\begin{array}{ccccc|c} \boxed{1} & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & \boxed{1} & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\uparrow \qquad \uparrow$

dim Nul A = # free var = 3
dim Col A = # pivots = 2
columns = 5