4.7: Change of Basis

Recall:

V = vector space , $\mathcal{B}$ = basis of V

then for any $x \in V$, $[x]_{\mathcal{B}}$ is unique

and it is the "coordinates" of x in basis $\mathcal{B}$

New:

if $\mathcal{C}$ = a different basis of V

then how are $[x]_{\mathcal{B}}$ and $[x]_{\mathcal{C}}$ related?

see figure 1 , p272

Ex: $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ are bases of vector space V such that

$$b_1 = 2c_1 + 7c_2 \qquad b_2 = c_1 - 4c_2$$

and we know

$$x = 2b_1 + 3b_2$$

which means

$$[x]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \text{ find } [x]_{\mathcal{C}}$$

way 1:

$$\begin{aligned} x &= 2b_1 + 3b_2 \\ &= 2(2c_1 + 7c_2) + 3(c_1 - 4c_2) \\ &= 4c_1 + 14c_2 + 3c_1 - 12c_2 \\ &= 7c_1 + 2c_2 \end{aligned}$$

$$[x]_{\mathcal{C}} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

way 2:

$$[x]_{\mathcal{C}} = [2b_1 + 3b_2]_{\mathcal{C}} = 2[b_1]_{\mathcal{C}} + 3[b_2]_{\mathcal{C}}$$

$$= \underbrace{\begin{bmatrix} [b_1]_{\mathcal{C}} & [b_2]_{\mathcal{C}} \end{bmatrix}}_{\text{a matrix}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$[b_1]_C = \begin{bmatrix} 2 \\ 7 \end{bmatrix} \quad \text{and} \quad [b_2]_C = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$\Rightarrow \quad [x]_C = \begin{bmatrix} 2 & 1 \\ 7 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4+3 \\ 14-12 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

Thrm

Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ and $\mathcal{C} = \{c_1, \ldots, c_n\}$ be bases of a vector space V.

Then $\exists!$ $n \times n$ matrix $\underset{C \leftarrow \mathcal{B}}{P}$ $\ni$

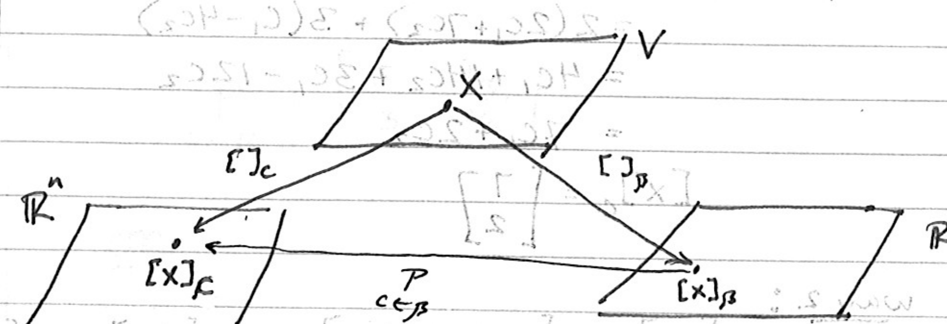
$$\boxed{[x]_C = \underset{C \leftarrow \mathcal{B}}{P} [x]_{\mathcal{B}}}$$

the columns of $\underset{C \leftarrow \mathcal{B}}{P}$ are the C-coordinates of the vectors in the basis $\mathcal{B}$

ie: $\underset{C \leftarrow \mathcal{B}}{P} = \big[[b_1]_C \; [b_2]_C \cdots [b_n]_C \big]$

Def: Change of Coordinates Matrix (from $\mathcal{B}$ to C)

$\underset{C \leftarrow \mathcal{B}}{P}$ is this matrix which changes your $\mathcal{B}$-coordinates into C-coordinates



$$[x]_{\mathcal{B}} = \Big(\underset{C \leftarrow \mathcal{B}}{P} \Big)^{-1} [x]_C$$

$$\underset{\mathcal{B} \leftarrow C}{P} = \Big(\underset{C \leftarrow \mathcal{B}}{P} \Big)^{-1}$$

← notes backwards from here on

<u>ex</u>: Find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ where $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1; c_2\}$

$$b_1 = \begin{bmatrix} 7 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 4 \\ -5 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

now $\underset{C \leftarrow \mathcal{B}}{P} = \left[[b_1]_c \; [b_2]_c \right]$ so we need to

find the C-coordinates of $\mathcal{B}$

b_1:

$$x_1 \begin{bmatrix} 4 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

or $$\begin{bmatrix} 4 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad [b_1]_c = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$$

b_2:

$$y_1 \begin{bmatrix} 4 \\ -5 \end{bmatrix} + y_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

or $$\begin{bmatrix} 4 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad [b_2]_c = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 2/6 \end{bmatrix}$$

$$\underset{C \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 5 & -1/3 \\ 13 & 2/6 \end{bmatrix}$$

<u>OR</u>

$$\left[\bar{c}_1 \; \bar{c}_2 \;\middle|\; \bar{b}_1 \; \bar{b}_2 \right] \sim \left[I \;\middle|\; \underset{C \leftarrow \mathcal{B}}{P} \right]$$

$\leftarrow$

ex: $b_1 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $c_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $c_2 = \begin{bmatrix} -7 \\ 3 \end{bmatrix}$

then $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ are basis for $\mathbb{R}^2$.

a) Find the change of coordinate matrix from $\mathcal{C}$ to $\mathcal{B}$

b) Find the change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$

(a)

$$\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \left[b_1 \; b_2 \,|\, c_1 \; c_2 \right] = \left[\begin{array}{cc|cc} 4 & 1 & 5 & -7 \\ -3 & -2 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{\text{rref}} \left[\begin{array}{cc|cc} 1 & 0 & 2.2 & -2.2 \\ 0 & 1 & -3.8 & 1.8 \end{array}\right]$$

$$\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \begin{bmatrix} 2.2 & -2.2 \\ -3.8 & 1.8 \end{bmatrix}$$

(b)

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left(\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}\right)^{-1} = \frac{1}{-4.4} \begin{bmatrix} 1.8 & 2.2 \\ 3.8 & 2.2 \end{bmatrix}$$

$$= \begin{bmatrix} -9/22 & -1/2 \\ -19/22 & -1/2 \end{bmatrix}$$

Note:

recall $P_{\mathcal{B}}$ converted $\mathcal{B} \rightarrow$ std coordinates

& $P_{\mathcal{C}}$ converted $\mathcal{C} \rightarrow$ std coordinates

then

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = P_{\mathcal{C}}^{-1} \cdot P_{\mathcal{B}}$$

$\mathcal{B}$-coordin. to std coor.

std coordin. to $\mathcal{C}$-coordin.

$\mathcal{B}$-coordin. to $\mathcal{C}$-coordin.