5.1: Eigenvectors & Eigenvalues

Recall transformations of the form: $x \mapsto Ax$

ex: $A = \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $x = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$$Av = \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix} = 3v$$

for v, $v \longmapsto 3v$ (stretches)

$$Ax = \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \end{bmatrix} = -x$$

for x, $x \longmapsto -x$ (reverses direction)

* these are special vectors for the matrix A (where A just gives back a multiple of the vector)

Def: Eigenvector

an eigenvector of an $n \times n$ matrix A is a vector x where $Ax = \lambda x$ and λ is just some constant

Def: Eigenvalue

λ, a constant, is an eigenvalue of A if $Ax = \lambda x$ has a non-trivial solution x

note: that x is the cooresponding eigenvector of λ

$$A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}$$

ex: are $\bar{u}$ & $\bar{v}$ eigenvectors of A?

$$u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

$$A\bar{u} = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 16-4 \\ -12+18 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix} = 3\bar{u} \checkmark$$

$$A\bar{v} = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -8-10 \\ 6+45 \end{bmatrix} = \begin{bmatrix} -18 \\ 51 \end{bmatrix} \neq \lambda\bar{v}$$

ex: Is $\lambda = 10$ an eigenvalue of A?

$$A\bar{x} = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \overset{?}{=} \begin{bmatrix} 10x \\ 10y \end{bmatrix} = \lambda\bar{x}$$

$$\Longrightarrow 4x - 2y = 10x$$
$$0 = 6x + 2y$$
$$y = -3x$$
$$-3x + 9y = 10y$$
$$-3x = y$$

So, pick $x = 1 \Longrightarrow y = -3$

$$A\bar{x} = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 4+6 \\ -3-27 \end{bmatrix} = \begin{bmatrix} 10 \\ -30 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

yes, $\lambda = 10$ is an eigenvalue since we found an $\bar{x}$

the collection of all such $\bar{x}$'s is called an eigenspace.

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Finding the "eigenspace" of our example for $\lambda = 10$

$A\bar{x} = 10\bar{x}$ happens when $\bar{x} = \begin{bmatrix} a \\ b \end{bmatrix}$

and we have the relation:

$$b = -3a$$

or $\bar{x} = \begin{bmatrix} a \\ -3a \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} a$

eigenspace of A relative to $\lambda = 10$ is span $\left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$

the basis of this eigenspace is $\left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$

__Thrm__

eigenvalues of a triangular matrix are the values on the diagonal

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If $v_1, \ldots, v_r$ are eigenvectors cooresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ then the set $\{v_1, \ldots, v_r\}$ is Linearly independent.

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