5.3: Diagonalization

Want: $A = PDP^{-1}$

goal $\Longrightarrow$ find P and D

Example: $D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$

$$D^2 = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} (-2)^2 & 0 \\ 0 & 5^2 \end{bmatrix}$$

$$D^3 = DD^2 = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} (-2)^2 & 0 \\ 0 & 5^2 \end{bmatrix} = \begin{bmatrix} (-2)^3 & 0 \\ 0 & 5^3 \end{bmatrix}$$

$$\Rightarrow D^k = \begin{bmatrix} (-2)^k & 0 \\ 0 & 5^3 \end{bmatrix}$$

Example: $A = \begin{bmatrix} 12 & 14 \\ -7 & -9 \end{bmatrix}$ Find A^k given $A = PDP^{-1}$

where $D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$$
$$= \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} (-2)^2 & 0 \\ 0 & 5^2 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$$

$$A^3 = (PDP^{-1})A^2 = PDP^{-1}PD^2P^{-1} = PD^3P^{-1}$$

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} (-2)^k & 0 \\ 0 & 5^k \end{bmatrix}\begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -(-2)^k + 2\cdot 5^k & (-2)^{k+1} + 2\cdot 5^k \\ (-2)^k - 5^k & -(-2)^{k+1} - 5^k \end{bmatrix}$$

$\leftarrow$

Diagonalizable

Matrix A (nxn) is diagonalizable if we can find P & D, where P is invertible and D is a diagonal matrix, such that

$$A = PDP^{-1}$$

The Diagonalization Thm

an nxn matrix A is diagonalizable iff A has n Linearly independent eigenvectors

note: the eigenvalues of A are what we place along the diagonal of D

Example: $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$. if possible, diagonalize A.

Step 1: Find the eigenvalues of A

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix}$$

$$0 = (2-\lambda)(1-\lambda) - 12 = \lambda^2 - 3\lambda - 10$$

$$\Longrightarrow 0 = (\lambda - 5)(\lambda + 2)$$

So $\lambda = 5, -2$

Step 2: Find 2 Lin. Indep. eigenvectors of A

$\lambda = 5 \Longrightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} = v_1$

$\lambda = -2 \Longrightarrow \begin{bmatrix} 3 \\ -4 \end{bmatrix} = v_2$

$\leftarrow$

step 3 : Make P

$$P = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix}$$

Step 4 : make D

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\therefore \; A = PDP^{-1}$$

Example

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

step 1 :

$$0 = \begin{vmatrix} 4-\lambda & 0 & -2 \\ 2 & 5-\lambda & 4 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (4-\lambda)(5-\lambda)^2$$

$$\Rightarrow \lambda = 4, 5$$

multi = 2

Step 2 :

$\lambda = 4$ $\quad A \cdot \bar{x} = 4\bar{x} \quad \Rightarrow \quad z = 0$ & $y = -2x$

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$\lambda = 5$ $\quad A\bar{x} = 5\bar{x} \quad \Rightarrow \quad x = -2z$

$$v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \text{ and } v_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

←

Step 3

$$P = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Step 4

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$\therefore \; A = PDP^{-1}$

Thm
an $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Example: Is A diagonalizable?

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

the eigenvalues of a triangular matrix is $\lambda = 4, -3, 1$

$\Rightarrow$ 3 distinct eigenvalues

So A is diagonalizable

Non-distinct Eigenvalues

A is $n \times n$

if A has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$
with cooresponding eigenvectors $v_1, \dots, v_2$
$\Rightarrow P = [v_1 \; v_2 \dots v_n]$ is automatically invertible

if A has $r < n$ distinct eigenvalues $\lambda_1, \dots, \lambda_r$
& if A is diagonalizable, we can still build P to be invertible

<u>thm</u>:

A is $n \times n$ and its distinct eigenvalues are $\lambda_1, \dots, \lambda_r$ where $r < n$

(a) for k (where $1 \le k \le r$), the dimension of the eigenspace of λ_k is less than or equal to the multiplicity of λ_k

(b) A is diagonalizable iff the sum of the dimensions of the distinct eigenspaces equals n. Note, this will only happen if dimension of the eigenspace of λ_k is the multiplicity of λ_k (for each distinct eigenvalue of A)

(c) If A diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace of λ_k for each k, then all the vectors in $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_r$ form an eigenvector basis of $\mathbb{R}^n$

←

ex: Diagonalize $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$

1) eigenvalues are $\lambda = 5, -3$

2) $\lambda = 5$: $A\bar{x} = 5\bar{x}$

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 5a \\ 5b \\ 5c \\ 5d \end{bmatrix}$$

$$\begin{aligned} 5a &= 5a \\ 5b &= 5b \\ a+4b-3c &= 5c \\ -a-2b-3d &= 5d \end{aligned}$$

$$\Rightarrow \begin{cases} a+4b = 8c \\ a+2b = -8d \end{cases}$$

$$\Rightarrow \begin{cases} a = -2b-8d \\ 2b = 8c+8d \end{cases}$$

$$\Rightarrow \begin{cases} a = -8c-16d \\ b = 4c+4d \end{cases}$$

make sure they're Linearly independent

in parametric vector form:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -8c-16d \\ 4c+4d \\ c \\ d \end{bmatrix} = \underbrace{\begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix}}_{= v_1} c + \underbrace{\begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix}}_{= v_2} d$$

$\underline{\underline{\lambda = -3}}$ $A\bar{x} = -3\bar{x}$

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -3a \\ -3b \\ -3c \\ -3d \end{bmatrix}$$

$$\begin{cases} 5a = -3a & \Rightarrow a = 0 \\ 5b = -3b & \Rightarrow b = 0 \\ a + 4b - 3c = -3c & \Rightarrow -3c = -3c \\ -a - 2b - 3d = -3d & \Rightarrow -3d = -3d \end{cases}$$

$\boxed{a = b = 0}$

and c & d can be any thing

$$V_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad V_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

3) $P = [V_1\ V_2\ V_3\ V_4]$

$$P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

4)
$$D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Conclusión: $A = PDP^{-1}$

$\leftarrow$ Application: Compute A^{20}