transformation: $T(\bar{x}) = A\bar{x}$ <u>or</u> $\bar{x} \mapsto A\bar{x}$

is essentially the same as $u \mapsto D\bar{u}$,

where $A = PDP^{-1}$

<u>Recall</u>:

$T: \mathbb{R}^n \to \mathbb{R}^m$ Linear transformation

can be written as $T(\bar{x}) = A\bar{x}$

where A = standard matrix of T

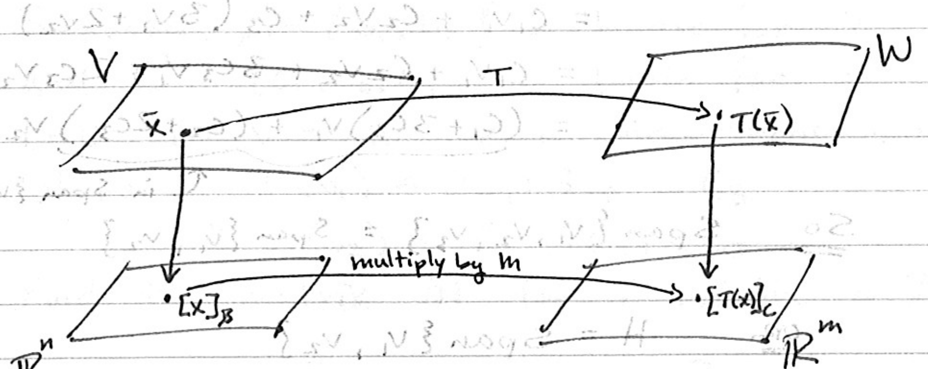
* $A = [T(e_1) \ \dots \ T(e_n)]$

Matrix of a Linear Transformation

$\dim V = n$ * $\dim W = m$

and

$T: V \to W$ Linear transformation



$\mathcal{B}$ is a basis for V (ie: $\mathcal{B} = \{b_1, \dots, b_n\}$)

C is a basis for W (ie: $C = \{c_1, \dots, c_m\}$)

we can write

$\bar{x} = r_1\bar{b}_1 + r_2\bar{b}_2 + \dots + r_n\bar{b}_n$

←

So, $[x]_\beta = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$

and

$$T(\bar{x}) = T(r_1 b_1 + \dots + r_n \bar{b}_n) = r_1 T(b_1) + \dots + r_n T(b_n)$$

but $T(\bar{x}) \in W$ so we can write in terms of C

$$[T(x)]_C = r_1 [T(b_1)]_C + \dots + r_n [T(b_n)]_C$$

$\Longrightarrow$

$$\boxed{[T(\bar{x})]_C = M \cdot [x]_\beta}$$

where

$$M = \big[[T(b_1)]_C \ \dots \ [T(b_n)]_C \big]$$

M is the matrix for T relative to the bases $\mathcal{B}$ & C

<u>ex:</u> Suppose $\mathcal{B} = \{b_1, b_2\}$ is basis for V
and $C = \{c_1, c_2, c_3\}$ is basis for W
Let $T: V \to W$ be a Linear transformation
where $T(b_1) = 5c_1 + 3c_2 - 2c_3$
& $T(b_2) = c_1 - 4c_2 + 2c_3$
Find the matrix M for T relative to $\mathcal{B}$ & C

$$M = \big[[T(b_1)]_C \ [T(b_2)]_C \big]$$

$$M = \begin{array}{cc} \begin{bmatrix} 5 & 1 \\ 3 & -4 \\ -2 & 2 \end{bmatrix} & \begin{array}{l} \leftarrow c_1 \\ \leftarrow c_2 \\ \leftarrow c_3 \end{array} \end{array}$$

$\leftarrow$

Linear Transformations from V into V

(when $V = W$) and (when $\mathcal{B} = \mathcal{C}$)
then

matrix M is called the matrix for T relative to $\mathcal{B}$
or the $\mathcal{B}$-matrix for T (ie: $M = [T]_{\mathcal{B}}$)

for $T : V \to V$ ~~where~~ we get

$$[T(x)]_{\mathcal{B}} = [T]_{\mathcal{B}} \cdot [x]_{\mathcal{B}} \quad , \forall x \in V$$

ex: $T : \mathbb{P}_3 \to \mathbb{P}_3$ is a Linear Transformation defined by $T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = a_1 + 2a_2 t + 3a_3 t^2$

a) Find the $\mathcal{B}$-matrix for T, $\mathcal{B} = \{1, t, t^2, t^3\}$

b) Verify $[T(\bar{p})]_{\mathcal{B}} = [T]_{\mathcal{B}} [\bar{p}]_{\mathcal{B}}$, $\forall p \in \mathbb{P}_3$

ⓐ $$M = [T]_{\mathcal{B}} = \Big[[T(1)]_{\mathcal{B}} \; [T(t)]_{\mathcal{B}} \; [T(t^2)]_{\mathcal{B}} \; [T(t^3)]_{\mathcal{B}} \Big]$$

$$= \begin{array}{cc} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{l} \leftarrow 1 \\ \leftarrow t \\ \leftarrow t^2 \\ \leftarrow t^3 \end{array} \end{array}$$

ⓑ $$[T]_{\mathcal{B}} [\bar{p}]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{bmatrix} = [T(\bar{p})]_{\mathcal{B}} \quad \checkmark$$

←

Linear Transformations on $\mathbb{R}^n$

note: if A diagonalizable
then $\exists$ a basis $\mathcal{B}$ for $\mathbb{R}^n$ made of
up eigenvectors

Diagonal Matrix Representation

Suppose $A = PDP^{-1}$, D is diagonal $n \times n$ matrix
If $\mathcal{B}$ is basis for $\mathbb{R}^n$ formed from the
columns of P
Then D is the $\mathcal{B}$-matrix for the
transformation $\bar{x} \mapsto A\bar{x}$

ex: $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix}^{-1}$

Def $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\bar{x}) = A\bar{x}$
Find a basis $\mathcal{B}$ where $[T]_{\mathcal{B}}$ is diagonal.

$\mathcal{B} = \{b_1, b_2\}$ where
$b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

then $[T]_{\mathcal{B}} = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$

$\leftarrow$

Similarity of Matrix Representation

note if $A = PCP^{-1}$

and if $\mathcal{B}$ = columns of P

<u>then</u> C is the $\mathcal{B}$-matrix of T
where $T(\bar{x}) = A\bar{x}$

<u>ex</u>: $A = \begin{bmatrix} 4 & -9 \\ 4 & 8 \end{bmatrix}$, $b_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $b_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

characteristic poly of A is $(\lambda+2)^2 = 0$
$\Rightarrow$ only $\lambda = -2$
but eigenspace of -2 is 1-dim
(we can't find 2 eigenvectors)
so, not diagonalizable

<u>but</u> for $\mathcal{B} = \{b_1, b_2\}$ the $\mathcal{B}$-matrix of
$T: \mathbb{R}^2 \to \mathbb{R}^2$ where $T(\bar{x}) = A\bar{x}$
is triangular

$A = PCP^{-1}$ $(C = P^{-1}AP)$

$[T]_{\mathcal{B}} = C = P^{-1}AP$

$= \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$

called Jordan Normal Form