Onto / one-to-one

$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by $T(\bar{x}) = A\bar{x}$

So A is $m \times n$

$\begin{cases} m = \text{\# of rows} \\ n = \text{\# of columns} \end{cases}$

<u>Theorem</u>:

1. $T(\bar{x}) = A\bar{x}$ is onto if and only if the columns of A span $\mathbb{R}$

   means we need at least m columns (ie: $n \geq m$)

2. $T(\bar{x}) = A\bar{x}$ is one-to-one if and only if the columns of A are linearly independent.

   means we can't have any more than m columns (ie: $n \leq m$)

So, for T to be both 1-to-1 and onto, we need $n = m$ <u>or</u> we must have a square matrix.

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3 situations:

$\boxed{n \leq m}$ $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \Longrightarrow$ columns are $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

these vectors can't span $\mathbb{R}^3$ since ~~they're~~ there's only 2 of them, but they ~~could be~~ are linearly independent (ie: not onto ~~maybe~~ but ~~are~~ is 1-1)

$\boxed{n = m}$ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Longrightarrow$ columns are $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

these vectors might be able span $\mathbb{R}^2$ & are linearly independent (ie: 1-1 & onto)

$\boxed{n \geq m}$

$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \Longrightarrow$ columns: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

not linearly independent, might span $\mathbb{R}$

so not 1-to-1 maybe onto