Practice Homework Question:

Suppose $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible.
Explain why T is both one-to-one and onto.

Solution:

(a) one-to-one means there exists only one vector $\bar{x} \in \mathbb{R}^n$ such that $T(\bar{x}) = \bar{y}$ for some particular $\bar{y} \in \text{Image}(T)$. Also, T is invertible means there is a linear transformation $S: \mathbb{R}^n \to \mathbb{R}^n$ where $S(T(\bar{x})) = \bar{x} = T(S(\bar{x}))$. There are 2 ways to give a formal proof that T is one-to-one:

way 1:

suppose we have 2 vectors $\bar{x}$ and $\bar{y}$ in $\mathbb{R}^n$ where $T(\bar{x}) = T(\bar{y})$. No assumption that $\bar{x} \neq \bar{y}$ or if $\bar{x} = \bar{y}$.

then we apply $S(\cdot)$ to both sides:

$$S(T(\bar{x})) = S(T(\bar{y}))$$
$$\Rightarrow \quad \bar{x} = \bar{y}$$

thus, if $T(\bar{x}) = T(\bar{y})$ for any $\bar{x}, \bar{y} \in \mathbb{R}^n$ we get that $\bar{x} = \bar{y}$

this is the definition that T is 1-1

way 2: (called proof by contradiction)

Suppose we have $\bar{x}, \bar{y} \in \mathbb{R}^n$ such that $T(\bar{x}) = T(\bar{y})$

Assume that $\bar{x} \neq \bar{y}$

we apply $S(\cdot)$ to both sides to get:

$$S(T(\bar{x})) = S(T(\bar{y}))$$
$$\Rightarrow \quad \bar{x} = \bar{y}$$

but this is a contradiction of our above assumption

thus, the assumption must be wrong

so $\bar{x} = \bar{y}$

thus, T is 1-1

(b) onto means there exists some vector $\bar{x} \in \mathbb{R}^n$ such that for every $\bar{y}$ in $\mathbb{R}^n$ we have $T(\bar{x}) = \bar{y}$

---

so we pick any $\bar{y}$ in $\mathbb{R}^n$

now we need to find some $\bar{x}$ in $\mathbb{R}^n$ where $T(\bar{x}) = \bar{y}$

Let $\bar{x} = S(\bar{y})$

check if this $\bar{x}$ works:

$$T(\bar{x}) = T(S(\bar{y})) = \bar{y}$$

So since $\bar{y}$ was arbitrary, we get that for any vector in $\mathbb{R}^n$ we can find another vector in $\mathbb{R}^n$ such that $T(\bar{x}) = \bar{y}$

$\Rightarrow T$ is onto

Note: the key that we used for both of these proofs was that we had the inverse transformation of T, namely we had $S(\cdot)$.