Theorems and Corollaries of Chapter 6

Theorem 6.1: A non-empty subset I of a ring R is an ideal if and only if it has the properties:

i. if $a, b \in I$, then $a - b \in I$;
ii. if $r \in R$ and $a \in I$, then $ra \in I$ and $ar \in I$.

Theorem 6.2: Let R be a commutative ring with identity, c an element of R, and I the set of all multiples of c in R, that is, $I = \{rc \mid r \in R\}$. Then I is an ideal.

Theorem 6.3: Let R be a commutative ring with identity and $c_1, c_2, \dots, c_n \in R$. Then the set $I = \{r_1c_1 + r_2c_2 + \cdots + r_nc_n \mid r_1, r_2, \dots, r_n \in R\}$ is an ideal in R.

Theorem 6.4: Let I be an ideal in a ring R. Then the relation of the congruence modulo I is

i. reflexive: $a \equiv a \; mod \; I$ for every a in R;
ii. symmetric: if $a \equiv b \; mod \; I$, then $b \equiv a \; mod \; I$;
iii. transitive: if $a \equiv b \; mod \; I$ and $b \equiv c \; mod \; I$, then $a \equiv c \; mod \; I$.

Theorem 6.5: Let I be an ideal in a ring R. If $a \equiv b \; mod \; I$ and $c \equiv d \; mod \; I$, then

i. $a + c \equiv b + d \; mod \; I$;
ii. $ac \equiv bd \; mod \; I$.

Theorem 6.6: Let I be an ideal in a ring R and $a, b \in R$. Then $a \equiv b \; mod \; I$ if and only if $a + I = b + I$.

Corollary 6.7: Let I be an ideal in a ring R. Then two cosets of I are either disjoint or identical.

Theorem 6.8: Let I be an ideal in a ring R. If $a + I = b + I$ and $c + I = d + I$ in R/I, then

$$(a + c) + I = (b + d) + I \quad \text{and} \quad ac + I = bd + I$$

Theorem 6.9: Let I be an ideal in the ring R. Then

i. R/I is a ring with addition and multiplication of cosets as defined in Theorem 6.8.
ii. If R is commutative, then R/I is a commutative ring.
iii. If R has an identity, then so does the ring R/I.

Theorems and Corollaries of Chapter 6

Theorem 6.10: Let $f: R \to S$ be a homomorphism of rings, and let $K = \{r \in R \mid f(r) = 0_S\}$. Then K is an ideal in the ring R.

Theorem 6.11: Let $f: R \to S$ be a homomorphism of rings with kernel K. Then $K = \langle 0_K \rangle$ if and only if f is injective.

Theorem 6.12: Let I be an ideal in a ring R. Then the map $\pi: R \to R/I$ given by $\pi(r) = r + I$ is a surjective homomorphism with kernel I.

Theorem 6.13 (First Homomorphism Theorem): Let $f: R \to S$ be a surjective homomorphism of rings with kernel K. Then the quotient ring R/K is isomorphic to S.

Second Homomorphism Theorem: Let I and J be ideals in ring R. Then $I \cap J$ is an ideal in I, and J is an ideal in $I + J$. Then $\frac{I}{I \cap J} = \frac{I+J}{J}$.

Third Homomorphism Theorem: Let I and K be ideals in ring R such that $K \subseteq I$, so I/K is an ideal in R/K. Then $\frac{(R/k)}{(I/K)} \cong R/I$.

Theorem 6.14: Let P be an ideal in a commutative ring R with identity. Then P is prime if and only if the quotient ring R/I is an integral domain.

Theorem 6.15: Let M be an ideal in a commutative ring R with identity. Then M is a maximal ideal if and only if the quotient ring R/M is a field.

Corollary 6.16: In a commutative ring R with identity, every maximal ideal is prime.