due April 3 Homework Set 11 (section 4.4) Name: _____

When writing a proof, be sure to cite all of the properties, theorems, corollaries, and definitions you use. Be sure to write all of your answers in complete sentences (even the non-proof questions).

1. Find the remainder when $f(x)$ is divided by $g(x)$.
   a. $f(x) = 2x^5 - 3x^4 + x^3 + 2x + 3$ and $g(x) = x - 3$ in $\mathbb{Z}_5[x]$

   b. $f(x) = 10x^{75} - 8x^{65} + 6x^{45} + 4x^{37} - 2x^{15} + 5$ and $g(x) = x + 1$ in $\mathbb{Q}[x]$

2. For what value of k is $x + 1$ a factor of $x^4 + 2x^3 - 3x^2 + kx + 1$ in $\mathbb{Z}_5[x]$?

3. Let F be a field. If $f(x)$ and $g(x)$ are associates in $F[x]$, prove that the two polynomials have the same roots in F.

4. Find the smallest prime $p > 5$ such that $x^2 + 1$ is reducible in $\mathbb{Z}_p[x]$.

5. $\mathbb{Q}[\sqrt{2}]$ is the set of numbers of the form $r_0 + r_1\sqrt{2} + r_2(\sqrt{2})^2 + \cdots + r_n(\sqrt{2})^n$ where n is a non-negative integer and all of the $r_i \in \mathbb{Q}$.

    a. Prove that $\mathbb{Q}[\sqrt{2}]$ is a subring of $\mathbb{R}$.

    b. Prove that the function $\varphi: \mathbb{Q}[x] \to \mathbb{Q}[\sqrt{2}]$ defined by $\varphi(f(x)) = f(\sqrt{2})$ is a surjective homomorphism, but not an isomorphism.