Theorems and Corollaries of Chapter 3

Theorem 3.1: Let R and S be rings. Define addition and multiplication on the Cartesian Product $R \times S$ by $(r, s) + (r', s') = (r + r', s + s')$ and $(r, s)(r', s') = (rr', ss')$. Then $R \times S$ is a ring. If both R and S are commutative, then so is $R \times S$. If both R and S have an identity, then so does $R \times S$.

Theorem 3.2: Suppose that R is a ring and that S is a subset of R such that

i. S is closed under addition (ie: if $a, b \in S$, then $a + b \in S$);
ii. S is closed under multiplication (ie: if $a, b \in S$, then $ab \in S$);
iii. 0_R is in S; and
iv. If $a \in S$, then the solution to the equation $a + x = 0_R$ is in S.

Then S is a subring of R.

Theorem 3.3: For any element a in a ring R, the equation $a + x = 0_R$ has a unique solution.

Theorem 3.4: If $a + b = a + c$ in ring R, then $b = c$.

Theorem 3.5: For any elements a and b in a ring R,

1. $a \cdot 0_R = 0_R = 0_R \cdot a$.
2. $a(-b) = -ab = (-a)b$.
3. $-(-a) = a$.
4. $-(a + b) = (-a) + (-b)$.
5. $-(a - b) = (-a) + b$.
6. $(-a)(-b) = ab$.

If R has an identity, then

7. $(-1_R)a = -a$.

Theorem 3.6: Let S be a nonempty subset of a ring R such that

i. S is closed under subtraction (ie: if $a, b \in S$, then $a - b \in S$); and
ii. S is closed under multiplication.

Then S is a subring of R.

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Theorem 3.7: Let R be a ring, and let a and b be in R. Then the equation $a + x = b$ has the unique solution $x = b - a$.

Theorem 3.8: Let R be a ring with identity and a and b elements of R. If a is a unit, then each of the equations $ax = b$ and $ya = b$ has a unique solution in R.

Theorem 3.9: Every field is an integral domain.

Theorem 3.10: Cancellation is valid in any integral domain R (ie: if $a \neq 0_R$ and $ab = ac$ in R, then $b = c$).

Theorem 3.11: Every finite integral domain is a field.

Theorem 3.12: Let $f: R \rightarrow S$ be a homomorphism of rings. Then

1. $f(0_R) = 0_S$.
2. $f(-a) = -f(a)$ for every a in R.
3. $f(a - b) = f(a) - f(b)$ for all a and b in R.

If R is a ring with identity and f is surjective, then

4. S is a ring with identity, and $f(1_R) = 1_S$.
5. Whenever u is a unit in R, then $f(u)$ is a unit in S and $f(u)^{-1} = f(u^{-1})$.

Corollary 3.13: If $f: R \rightarrow S$ is a homomorphism of rings, then the image of f is a subring of S.