In class examples
Jan 24, 2013

1. Let a and b be integers. Prove that $\gcd(a, a+b) = \gcd(a, b)$.

Let $a, b \in \mathbb{Z}$.
By Theorem 1.3, there are integers u and v such that

$$\gcd(a, a+b) = au + (a+b)v = au + av + bv$$
$$= a(u+v) + bv \geq \gcd(a,b).$$

Also by Theorem 1.3, there are integers x and y such that

$$\gcd(a,b) = ax + by = ax - ay + ay + by$$
$$= a(x-y) + (a+b)y \geq \gcd(a, a+b).$$

Thus, $\gcd(a,b) = \gcd(a, a+b)$.

2. Let a and b be integers. Prove there exist integers t and w such that $ab = a^2t + b^2w$.

Let $a, b \in \mathbb{Z}$.
Suppose $d = \gcd(a,b)$. By definition, $\exists\, r, s \in \mathbb{Z} \ni$
$a = dr$ and $b = ds$. By Theorem 1.3, there are $u, v \in \mathbb{Z}$
where $d = au + bv$.

So, $ab = (dr)(ds) = d(r(ds)) = (au+bv)(r(ds))$
$= (au)(r(ds)) + (bv)(r(ds)) = (au)((rd)s) + (bv)(rb)$
$= (au)((dr)s) + (bv)(br) = (au)(as) + b(v(br))$
$= a(u(as)) + b((vb)r) = a((ua)s) + b((bv)r)$
$= a((au)s) + b(b(vr)) = a(a(us)) + b^2(vr)$
$= a^2(us) + b^2(vr).$

Set $t = us$ and $w = vr$.
Thus, $ab = a^2t + b^2w$.