due April 30

Homework Set 13
(section 5.3 – 6.2)

Name: ____________

When writing a proof, be sure to cite all of the properties, theorems, corollaries, and definitions you use. Be sure to write all of your answers in complete sentences (even the non-proof questions).

1. Answer the following (non-proof) questions:
   a. Determine whether the given ring is a field
      i. $\mathbb{Z}_3[x]/\langle x^3 + 2x^2 + x + 1\rangle$
      ii. $\mathbb{Z}_5[x]/\langle 2x^3 - 4x^2 + 2x + 1\rangle$
      iii. $\mathbb{Z}_2[x]/\langle x^4 + x^2 + 1\rangle$
   b. List the distinct principal ideals of the ring $\mathbb{Z}_{12}$.
   c. Explain why $\langle 4,6\rangle = \langle 2\rangle$ in $\mathbb{Z}$.

2. Let I and J be ideals in R. Prove that the set $I + J = \{a + b \mid a \in I, b \in J\}$ is an ideal in R that contains both I and J.

3. If $\gcd(m, n) = 1$ in $\mathbb{Z}$, prove that $\langle m \rangle \cap \langle n \rangle$ is the ideal $\langle mn \rangle$.

4. Let I and K be ideals in a ring R, with $K \subseteq I$. Prove that $I/K = \{a + K \mid a \in I\}$ is an ideal of the quotient ring R/K.

5. Suppose that I and J are ideals of a ring R, and let $f: R \to R/I \times R/J$ be the function defined by $f(a) = (a + I, a + J)$.
   a. Prove that f is a homomorphism of rings.

   b. Is f surjective?

   c. What is the kernel of f?