due Jan 24 — Homework Set 2 (section 1.2) — Name: _____________

1. Prove the Divisor Theorem part (6):
   Let b and d be integers with $d > 0$. If $b = dt$ for some integer t and $1 < d < b$, then $1 < t < b$.

Let b and d be integers with $d>0$. Suppose $b=dt$ for some integer t and $1<d<b$. Since $1<d$, we get that $t = 1\cdot t < d\cdot t = b$.
Since $1<b$, we know that $1<dt$. Combining this with $1<d$ and since t is an integer, we get that $t>0$. Thus, $t\geq 1$.
Now since $d \lneq b$, t cannot equal 1. Thus, combining all previous results yeilds $1<t<b$.

2. Prove the Divisor Theorem part (7):
   Let d be a positive integer. If $d = ds$ for some integer s, then $s = 1$.

Let d be a positive integer where $d=ds$ for some integer s.
Then $0=ds-d = d(s-1)$. Since d and $(s-1)$ are integers, we know that $d=0$ or $s-1=0$. However, by assumption, $d\neq 0$.
Thus, $s-1=0$, which is equivalent to $s=1$.

3. Let a be an integer. If 2 does not divide a, either prove or disprove that 4 divides $a^2 - 1$.

Let a be an integer such that 2 does not divide a. Hence, a is odd.
By section 1.1 exercise #8, which we proved in class on January 15, we may write $a^2 = 8k+1$ for some integer k.
Thus, $a^2-1 = 8k = 4(2k)$. By the definition, 4 divides a^2-1.

4. Let a, b, and c be integers. If a divides b and a divides c, prove that a divides $(br + ct)$ for any integer r and t.

Let a, b, and c be integers such that a divides both b and c.
By definition, $b = ak$ and $c = as$ for some integers k and s.
Let r and t be any integer, and consider:

$$br + ct = (ak)r + (as)t = a(kr) + a(st) = a(kr + st).$$

Thus, a divides $(br + ct)$.

5. Consider the set $A = \{36u + 42v \mid u, v \in \mathbb{Z}\}$.

a. How many elements are in the set A? Support your claim.

Set has infinitely many elements. Consider when $v = 0$. We can let u vary over all integers. Since $36u$ is different for every distinct u and there are infinitely many integers to choose from for u, we get that there are infinitely many elements in A.

b. List at least four distinct elements from the set A.

$36 = 36(1) + 42(0)$ $\quad$ $6 = 36(-1) + 42(1)$

$42 = 36(0) + 42(1)$ $\quad$ $0 = 36(0) + 42(0)$

c. Does set A have a smallest element? Why or why not.

There is no smallest element. If we let $v = 0$ as in part (a), we can always pick a more negative u. Hence, this means that $36u$ will get smaller and smaller.

d. Is there a smallest positive element? If so, state it. State why (or why not) we know there exists a smallest element.

The smallest positive element of A is 6. We know this element exists by the Well-ordering Axiom.

e. What is gcd(36,42)? In what way (if any) does this greatest common divisor relate to A?

The $\gcd(36, 42) = 6$. By Theorem 1.3, this is the smallest positive element in A.