Due March 28 | Homework Set 9 (Sections 3.3 – 4.1) | Name:_____________

Write all answers or solutions in complete, grammatically correct sentences. Be sure to cite all reasons.

1. Which of the following are homomorphisms? If the function is not a homomorphism, give an example of where it fails.
   a. $f: \mathbb{Z} \to \mathbb{Z}$, defined by $f(x) = -x$

   b. $g: \mathbb{Z}_2 \to \mathbb{Z}_2$, defined by $g(x) = -x$

   c. $h: \mathbb{Q} \to \mathbb{Q}$, defined by $h(x) = \frac{1}{x^2+1}$

   d. $k: \mathbb{Z}_{12} \to \mathbb{Z}_4$, defined by $k([x]_{12}) = [x]_4$ where $[u]_n$ is the equivalence class of u in $\mathbb{Z}_n$

   e. $D: \mathbb{R}[x] \to \mathbb{R}[x]$, defined by $D(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}$

2. If $g: R \to S$ and $f: S \to T$ are homomorphisms of rings, show that $f \circ g: R \to T$ is also a homomorphism of rings. If g and f are isomorphism, is $f \circ g$ also an isomorphism?

3. Let $f: R \to S$ be a homomorphism of rings, and consider $Ker(f) = \{r \in R \mid f(r) = 0_S\}$. Prove that $Ker(f)$ is a subring of R. (Note, $Ker(f)$ is called the kernel of f.)

4. Show that $\mathbb{Z}_2 \times \mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_8$.

5. Answer the following computational questions (no proof is necessary).
   a. Expand and simplify: $(x^2 - 3x + 2)(2x^3 - 4x + 1)$ in $\mathbb{Z}_7[x]$.

   b. Given polynomials $f(x) = 2x^4 + x^2 - x + 1$ and $g(x) = 2x - 1$, use the Division Algorithm to find polynomials $q(x)$ and $r(x)$ such that $f(x) = g(x)q(x) + r(x)$ where $\deg r(x) < \deg g(x)$ in $\mathbb{Z}_5$.

   c. List all polynomials of degree less than 3 in $\mathbb{Z}_3[x]$.