Supplement Exercises to MATH 5176

Section 1.2

1' Prove Theorem 2.2 in the reference materials [4].

Section 1.3

1' Find the truncation error of

$$\frac{v_m^{n+1} - v_m^n}{k} + a\left[\alpha \frac{3v_m^{n+1} - 4v_{m-1}^{n+1} + v_{m-2}^{n+1}}{2h} + (1-\alpha)\frac{3v_m^n - 4v_{m-1}^n + v_{m-2}^n}{2h}\right] = 0, \quad \alpha \in [0,1],$$

which is approximate to $\dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0$.

Section 2.1

1′ For a periodic funciton $f(x)$ which is piecewise continuious on $[-L, L]$ and has a left-hand derivative and right-hand derivative at each point of the interval $[-L, L]$, we have its Fourier series

$$f(x) = a_0 + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi}{L} x \right),$$

where

$$\begin{aligned} a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\, dx, \\ a_m &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L} dx, \\ b_m &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} dx, \\ & m = 1, 2, \cdots . \end{aligned}$$

Suppose that a function is defined on a grid: $x = mh,\ m = 0, \pm 1, \pm 2, \cdots$, and suppose the value of the funciton on $x = mh$ is v_m.

Based on the result on Fourier series, show that v_m can be expressed as

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \hat{v}(\xi)\, d\xi,$$

where

$$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-imh\xi} v_m h.$$

2' Show the Parseval relation $\|\hat{v}\|_h^2 = \|v\|_h^2$

Section 2.2

1' Show that for any l, the following holds:

$$\begin{aligned} \cos l\theta &= \sum_{j=0}^{l} e_{l,j} \sin^{2j} \frac{\theta}{2} \\ \sin l\theta &= \sin \frac{\theta}{2} \cos \frac{\theta}{2} \sum_{j=0}^{l-1} \bar{e}_{l,j} \sin^{2j} \frac{\theta}{2}, \end{aligned}$$

where $e_{l,j}$ and $\bar{e}_{l,j}$ are constants.

2' Necessary and sufficient conditions for

$$f_0 + f_1 \sin^2 \frac{\theta}{2} + f_2 \sin^4 \frac{\theta}{2} \geq 0$$

are

$$f_0 \geq 0, \quad 2f_0 + f_1 + 2\left[f_0\left(f_0 + f_1 + f_2\right)\right]^{\frac{1}{2}} \geq 0, \quad f_0 + f_1 + f_2 \geq 0.$$

3' If $\sum_l (-1)^{H_3(l)} d_l^* (\theta) d_l (\theta) \equiv 0$,

then the matrix

$$Q = \sum_l (-1)^{H_3(l)} D_l^* D_l$$

is a "pseudo-null" matrix. Here $H_3(l)$ is equal to either 0 or 1.

4' If $d_1^* (\theta) d_1 (\theta) - d_2^* (\theta) d_2 (\theta) \geq 0$, then the matrix $D_1^* D_2 - D_2^* D_2$ can be represented by a sum of one nonnegative definite matrix Z and pseudo-null matrix Q.

5' Consider

$$\frac{v_m^{n+1} - v_m^n}{k} + \frac{a}{2} \left[\frac{3v_m^{n+1} - 4v_{m-1}^{n+1} + v_{m-2}^{n+1}}{2h} + \frac{3v_m^n - 4v_{m-1}^n + v_{m-2}^n}{2h} \right] = 0.$$

In order for the Von Neumann condition to hold, what condition does $\frac{ak}{h}$ should satisfy?

6' Consider

$$\frac{v_m^{n+1} - v_m^n}{k} + a \left[\alpha \frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2h} + (1 - \alpha) \frac{v_{m+1}^n - v_{m-1}^n}{2h} \right] = 0.$$

In order for the Von Neumann condition to hold, for any $\alpha \in \left[0, \frac{1}{2}\right]$, what condition does $\frac{ak}{h}$ should satisfy? For any $\alpha \in \left[\frac{1}{2}, 1\right]$, what condition does $\frac{ak}{h}$ should satisfy?

7' Consider the inverse Lax-Friedichr scheme

$$\frac{\frac{1}{2} \left(v_{m+1}^{n+1} + v_{m-1}^{n+1} \right) - v_m^n}{k} + a \frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2h} = 0.$$

In order for the Von Neumann condition to hold, what condition does $\frac{ak}{h}$ should satisfy?

8' Using the Von Neumann analysis to show the stability condition of the scheme

$$\frac{v_m^{n+1} - v_m^n}{k} + \frac{a}{2} \left(\frac{v_{m+1}^{n+1} - v_m^{n+1}}{h} + \frac{v_m^n - v_{m-1}^n}{h} \right) = 0$$

and show that the truncation error is $O(k^2) + O(h^2)$.

8" Using the Von Neumann analysis to find the stability condition of the scheme

$$\frac{v_m^{n+1}-v_m^n}{k}+\frac{a}{2}\left(\frac{v_{m+1}^{n+1}-v_m^{n+1}}{h}+\frac{v_m^n-v_{m-1}^n}{h}\right)=0.$$

9' If the following holds:

$$T^n \leq (1+\bar{c}_0 h)\, T^{n+1}, \quad n=0,1,2,\cdots$$

and

$$\frac{1}{\bar{c}_1}\|v^n\|_h^2 \leq T^n \leq \bar{c}_1 \|v^n\|_h^2, \quad n=0,1,\cdots,$$

then we can prove that the scheme is stable.

10' Using the Theorem 1 in the reference material [3], prove that a horizontal 3-point scheme

$$\begin{aligned} & r_{-1m}^n v_{m-1}^{n+1}+r_{0m}^n v_m^{n+1}+r_{1,m}^n v_{m+1}^{n+1} \\ = \; & s_{-1m}^n v_{m-1}^n+s_{0,m}^n v_m^n+s_{1,m}^n v_{m+1}^n \end{aligned}$$

is stable, if the conditions (i)-(iv) in Theorem 1 hold.

11' Show that $a_1 \sin^2\frac{\theta}{2}+a_2\sin^4\frac{\theta}{2}+a_3\sin^6\frac{\theta}{2}$ can be written as

$$\sum_l d_l^*(\theta)\, m_l d_l(\theta)$$

and m_l will be positive numbers if

$$a_1 \geq 0, \;\; a_1+a_2+a_3 \geq 0, \;\; 2a_1+a_1+2\left[a_1(a_1+a_2+a_3)\right]^{\frac{1}{2}} \geq 0.$$

12' Show that a horizontal four-point explicit scheme with variable coefficients

$$v_m^{n+1} \;=\; s_{-1m}v_{m-1}^n+s_{0m}v_m^n+s_{1m}v_{m+1}^n+s_{2m}v_{m+2}^n, \quad (s_{im}=s_i(x_m))$$

is stable if the Von Neumann condition is satisfied at all the points and $\dfrac{\partial s_i(x)}{\partial x}$ and $\dfrac{\partial^2 s_i(x)}{\partial x^2}$ are continuous functions and bounded, $i=-1,0,1,2$.

13' Show $\left(1+\frac{1}{x}\right)^{ax} \leq e^a$ for any $x>0$ if $a>0$. (Hint: 1. show that $\left(1+\frac{1}{x}\right)^{ax}$ in an increasing function on $(0,\infty)$ if $a>0$. 2. If $f(a)>f(b)>0$ and $\frac{df}{dx}<0$ on $[a,b]$, then $f(x)>0$ on $[a,b]$.)

14' Show that when Von Neumann alaysis is used, replacing v_{m+k}^{n+k} by $g^{n+k}e^{i(m+j)\theta}$ and v_{m+k}^{n+k} by $g^k e^{ij\theta}$ will generate the same g.

15' Consider the scheme

$$\frac{\frac{v_{m+1}^{n+1}+v_m^{n+1}}{2}-\frac{v_{m+1}^{n}+v_m^{n}}{2}}{k}+a\frac{v_{m+1}^{n+1}-v_m^{n+1}+v_{m+1}^{n}+v_m^{n}}{2h}=0$$

and define $\lambda=\frac{k}{h}$. Find out the condition $a\lambda$ should satisfy in order for the scheme to be stable by using Von Neumann analysis.

16' Show that $f(x)=1+x-4a^2\lambda^2x^3\geq 0$ on $[0,1]$ is true if and only if $a^2\lambda^2\leq\frac{1}{2}$.

17' Find out when the scheme

$$\alpha v_{m+1}^{n+1}+(1-\alpha)\,v_{m-1}^{n+1}=v_m^n$$

is stable, and when it is unstable by using Von Neumann analysis.

18' Find the stability condition for each of the following scheme

(a)
$$\frac{v_m^{n+1}-v_m^n}{k}+a\frac{v_{m+1}^n-v_m^n}{h}=0;$$

(b)
$$\frac{v_m^{n+1}-v_m^n}{k}+a\frac{v_m^n-v_{m-1}^n}{h}=0;$$

(c)
$$\frac{v_m^{n+1}-v_m^n}{k}+a\frac{v_{m+1}^n-v_{m-1}^n}{2h}=0;$$

(d)
$$\frac{v_m^{n+1}-\frac{1}{2}\left(v_{m+1}^n+v_{m-1}^n\right)}{k}+a\frac{v_{m+1}^n-v_{m-1}^n}{2h}=0;$$

(e)
$$\begin{cases}\dfrac{\tilde{v}_m^{n+1}-v_m^n}{k}+\dfrac{v_{m+1}^n-v_{m-1}^n}{2h}=0,\\ v_m^{n+1}=\frac{1}{4}\left(\tilde{v}_{m+1}^{n+1}+2\tilde{v}_m^{n+1}+\tilde{v}_{m-1}^{n+1}\right).\end{cases}$$

Section 3.1

0' (a) Show

$$\frac{\partial u(t,x)}{\partial x} = \frac{3u(t,x) - 4u(t,x-h) + u(t,x-2h)}{2h} + 0\left(h^2\right),$$
$$\frac{\partial^2 u(t,x)}{\partial x^2} = \frac{u(t,x) - 2u(t,x-h) + u(t,x-2h)}{h^2} + 0(h).$$

(b) Show that for $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$

$$v_m^{n+1} = v_m^n - \frac{a\lambda}{2}\left(3v_m^n - 4v_{m-1}^n + v_{m-2}^n\right) + \frac{\alpha^2\lambda^2}{2}\left(v_m^n - 2v_{m-1}^n + v_{m-2}^n\right)$$

is a second order scheme.

0" Find the truncation error of

$$\frac{v_m^{n+1} - v_m^n}{k} + a\left[\alpha\frac{3v_m^{n+1} - 4v_{m-1}^{n+1} + v_{m-2}^{n+1}}{2h} + (1-\alpha)\frac{3v_m^n - 4v_{m-1}^n + v_{m-2}^n}{2h}\right] = 0, \quad \alpha \in [0,1],$$

which is approximate to $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$. (Hint: Use Taylor series at $t = t^{n+1/2}$ and $x = x_m$.)

1' **a** Consider $\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} = f(x,t)$. For this case, the MacCormack scheme given on page 77 can be rewritten as

$$v_m^{n+1} = v_m^n - \frac{a_m^n\lambda}{2}\left(v_{m+1}^n - v_{m-1}^n\right) + \frac{a_m^{n^2}\lambda^2}{2}\left(v_{m+1}^n - 2v_m^n + v_{m-1}^n\right) + \frac{k}{2}\left(f_m^{n+1} + f_m^n\right) - \frac{a_m^n k\lambda}{2}\left(f_m^n - f_{m-1}^n\right).$$

If $a(x,t) = constant$, it is a second order scheme. What is the truncation error of the scheme if $a(x,t)$ depends on x,t. If its truncation error is not $O\left(k^2\right) + O\left(h^2\right)$, derive a similar scheme whose truncation error is $O\left(k^2\right) + O\left(h^2\right)$.

b Replacing $a(x,t)$ by $a(u)$, study the problem mentioned in (a).

2' Show

$$\frac{u_m^{n+1} - u_{m-1}^n}{k} = \frac{\partial u_{m-\frac{1}{2}}^{n+\frac{1}{2}}}{\partial t} + \frac{h}{k}\frac{\partial u_{m-\frac{1}{2}}^{n+\frac{1}{2}}}{\partial x} + O\left(k^2\right) \text{ if } h = ck.$$

(Hint: Use Taylor series at $t = t^{n+1/2}$ and $x = x_{m-1/2}$.)

2" Consider $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$ with $a > 0$. For this case we have the following scheme

$$v_m^{n+\frac{1}{2}} = v_m^n - \frac{a\lambda}{2}\left(v_m^n - v_{m-1}^n\right)$$
$$v_m^{n+1} = v_{m-1}^n - (a\lambda - 1)\left(v_m^{n+\frac{1}{2}} - v_{m-1}^{n+\frac{1}{2}}\right).$$

Find its truncaiton error and the stability condition.

3' Show that the two-step scheme

$$\begin{aligned}\tilde{v}_m^{n+\frac{1}{2}} &= v_m^n - \frac{a_m^n\lambda}{2}\left(v_{m+1}^n - v_m^n\right) + \frac{k}{2}f_m^n \\ v_m^{n+1} &= v_{m-1}^n - \left(a_{m-\frac{1}{2}}^{n+\frac{1}{2}}\lambda - 1\right)\left(\tilde{v}_m^{n+\frac{1}{2}} - \tilde{v}_{m-1}^{n+\frac{1}{2}}\right) + kf_{m-\frac{1}{2}}^{n+\frac{1}{2}}\end{aligned}$$

is a second-order scheme approximating to

$$\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} = f(x,t)$$

and

$$\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = f(x,t) \quad \text{(In this case } a_m^n \text{ means } a(u_m^n)\text{)}.$$

3" Consider the scheme

$$\frac{v_m^{n+1} - v_m^n}{k} + \frac{a}{2}\left(\frac{v_{m+1}^{n+1} - v_m^{n+1}}{h} + \frac{v_m^n - v_{m-1}^n}{h}\right) = 0.$$

Show that its truncation error is $O(k^2) + O(h^2)$.

4' Prove Theorem 3.1.4 on page 71.

Section 3.2

1' (a) Modifying the MacCormack scheme given in the book (page 77), so that it is a second order scheme approximating to

$$\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} = f(x,t).$$

(b) Modifying the MacCormack scheme given in the book (page 77), so that it is a second order scheme approximating to

$$\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = f(x,t).$$

2' For each case given in Appendix A, determine whether or not the computation is stable and the order of the scheme by using the last two results if it is stable, and explain how the theoretical results support your conclusion.

3' For each of the two cases given in Appendix B, determine whether or not the computation is stable and whether or not the result is correct, and explain how the theoretical results support your conclusion.

4' Show that the Lax–Wendroff scheme is stable if and only if $|a\lambda| \le 1$.

5' Show that the Crank-Nicolson scheme is unconditionally stable.

Section 3.4

1' Derive the explicit one-sided second order scheme using a two-step method by which we derive the L-W scheme and analyze its stability condition.

2' Derive an implicit one-sided second order scheme to $\dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0$ in the form

$$\frac{v_m^{n+1} - v_m^n}{k} + \frac{a}{2h}\left[b_0 v_m^{n+1} + b_1 v_{m+1}^{n+1} + b_2 v_{m+2}^{n+1} + b_0 v_m^n + b_1 v_{m+1}^n + b_2 v_{m+2}^n\right] = 0,$$

show that such a scheme is unique and analyze its stability condition.

3' Derive an explicit one-sided second order scheme to $\dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0$ in the form

$$\frac{v_m^{n+1} - v_m^n}{k} + \frac{a}{2h}\left[b_0 v_m^n + b_1 v_{m+1}^n + b_2 v_{m+2}^n\right] = 0$$

and show that such a scheme is unique.

Section 3.5

1' **a.** Show that when the Crank-Nicolson scheme is used to solve a periodic problem, the system of equations can also be in the following form

$$\begin{pmatrix} b_1 & c_1 & 0 & \cdots & 0 & 0 & a_1 \\ a_2 & b_2 & c_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{m-1} & b_{m-1} & c_{m-1} \\ c_m & 0 & 0 & \cdots & 0 & a_m & b_m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_m \end{pmatrix}.$$

b. This system can also be solved by the following procedure: eliminate x_i from the all the systems and obtain

$$\alpha_{0,i+1}x_{i+1} + \beta_{0,i+1}x_{i+2} + \gamma_{0,i+1}x_m = h_{0,i+1},$$
$$\alpha_{1,i+1}x_{i+1} + \beta_{1,i+1}x_{m-1} + \gamma_{1,i+1}x_m = h_{1,i+1},$$

$i = 1, 2, \cdots, m-3$, then find x_m, x_{m-1}, x_{m-2}, and x_i, $i = m-3, m-4, \cdots, 1$, successively. Write down all the formulas for this procedure.

Section 4.1

1′ (a) Suppose that if g_+ and g_- are distinct roots of the equation

$$ag^2 + bg + c = 0,$$

then the set

$$x_n = c_1 g_+^n + c_2 g_-^n, \quad n = 0, 1, \ldots,$$

is a solution of the system

$$\begin{cases} ax_{n+1} + bx_n + cx_{n-1} = 0, \quad n = 1, 2, \cdots, \\ x_0 = c_1 + c_2, \\ x_1 = c_1 g_+ + c_2 g_-. \end{cases}$$

(b) Suppose that $\bar{g}$ is a double root of the equation

$$f(g) \equiv ag^2 + bg^1 + c = 0,$$

i.e.,

$$f(\bar{g}) = \frac{df(\bar{g})}{dg} = 0,$$

then

$$x_n = c_1 \bar{g}^n + c_2 n \bar{g}^{n-1}$$

is a soltuion of the system

$$\begin{cases} ax_{n+1} + bx_n + cx_{n-1} = 0, \quad n = 1, 2, \cdots, \\ x_0 = c_1, \\ x_1 = c_1 \bar{g} + c_2. \end{cases}$$

2′ (a) Suppose that $\bar{g}$ is a m-multiple root of the equation

$$f(g) \equiv a_l g^l + a_{l-1} g^{l-1} + \cdots + a_1 g + a_0 = 0,$$

i.e.,

$$f(\bar{g}) = \frac{df(\bar{g})}{dg} = \cdots = \frac{d^{m-1} f(\bar{g})}{dg^{m-1}} = 0,$$

then any of the functions

$$\begin{aligned} x_n &= \bar{g}^n, \\ x_n &= n\bar{g}^{n-1}, \\ &\cdots, \\ x_n &= n(n-1)\cdots(n-m+2)\bar{g}^{n-m+1} \end{aligned}$$

satisfies

$$a_l x_{n+1} + a_{l-1} x_n + \cdots + a_1 x_{n-l+2} + a_0 x_{n-l+1} = 0.$$

(Hint: First show $\left.\dfrac{d^k\left(g^{n+1-l} f(g)\right)}{dg^k}\right|_{g=\bar{g}} = 0,\ k = 1, 2, \cdots, m-1.$)

(b) Suppose that g_1, g_2, g_3 and g_4 are roots of the equation

$$f(g) = a_7 g^7 + a_6 g^6 + \cdots + a_1 g + a_0 = 0$$

with multiplicity 1, 1, 2, and 3 respectively, then for any $c_1, c_2, \cdots, c_7$,

$$\begin{aligned} x_n \;=\; & c_1 g_1^n + c_2 g_2^n + c_3 g_3^n + c_4 n g_3^{n-1} \\ & + c_5 g_4^n + c_6 n g_4^{n-1} + c_7 n(n-1) g_4^{n-2} \end{aligned}$$

satisfies

$$a_7 x_{n+1} + a_6 x_n + \cdots + a_1 x_{n-5} + a_0 x_{n-6} = 0.$$

3′ Consider the scheme

$$v_m^{n+1} + a_0 v_{m+1}^n + b_0 v_m^n + c_0 v_{m-1}^n + a_{-1} v_{m+1}^{n-1} + b_{-1} v_m^{n-1} + c_{-1} v_{m-1}^{n-1} = 0,$$
$$n = 1, 2, \cdots, \quad m = 0, \pm 1, \pm 2, \cdots,$$

and

$$v_m^1 + \bar{a}_0 v_{m+1}^0 + \bar{b}_0 v_m^0 + \bar{c}_0 v_{m-1}^0 = 0, \quad m = 0, \pm 1, \pm 2, \cdots,$$

with initial values v_m^0, $m = 0, \pm 1, \cdots$, where these coefficients in the scheme depend on k, k being a small positive number. Let $g_\pm(\theta)$ be two distinct roots of the equation

$$g^2 + g\left(a_0 e^{i\theta} + b_0 + c_0 e^{-i\theta}\right) + a_{-1} e^{i\theta} + b_{-1} + c_{-1} e^{-i\theta} = 0.$$

Show that if

$$\max_\theta |g_\pm(\theta)| \le 1 + \bar{c}k, \quad \bar{c} \text{ being a positive number independent of } k,$$

then this scheme is stable, i.e., for any n satisfying condition $nk \le T$, we have

$$||v^n||_h^2 \le c e^{\bar{c}T} ||v^0||_h^2, \quad c \text{ being a positive number.}$$

where

$$||v^n||_h^2 = h \sum_{m=-\infty}^{\infty} |v_m^n|^2, \quad n = 0, 1, \cdots, \quad h \text{ being a small positive number.}$$

Section 6.1

1′ Show that

$$u(t,x) = \frac{1}{\sqrt{t+\tau}} \exp\left(\frac{-(x-y)^2}{4b(t+\tau)}\right)$$

is a solution to the heat equation (6.1.1) for any values of y and τ.

2′ (a) Show that the solution of problem

$$\begin{cases} u_t + au_x = bu_{xx}, & -\infty < x < \infty, \quad t \geq 0, \\ u(0,x) = u_0(x), & -\infty < x < \infty \end{cases}$$

is

$$u(t,x) = \frac{1}{\sqrt{4\pi bt}} \int_{-\infty}^{\infty} \exp\left(\frac{-(x-y-at)^2}{4bt}\right) u_0(y)dy.$$

(Hint: Let $y = x - at$ and $u(t,x) = w(t,y)$, then show $w_t = bw_{yy}$ first.)

(b) Based on the result in a), show that if $t^* \geq t^{**}$, then the following inequality

$$\max_m |u(t^*,x)| \leq \max_m |u(t^{**},x)|$$

holds.

Section 6.3

1′ Find the truncation error of the Du Fort-Frankel scheme and analysis its stability. Explain why we cannot choose $k/h = \textit{constant}$ for computation.

Section 6.4

1′ Show that scheme (6.4.2) satisfies the condition $|g|^2 \leq 1$ if and only if $bk/h^2 < 1/2$ and $k \leq 2b/a^2$, and the condition $|g|^2 \leq 1 + ck$ if $bk/h^2 < 1/2$.

2′ Suppose $a > 0$. Show that scheme (6.4.7) satisfies the condition $|g|^2 \leq 1$ if and only if $2bk/h^2 + ak/h \leq 1$.

3′ Suppose $a > 0$. Show that the solutions of scheme (6.4.2) and scheme (6.4.7) have the property

$$\max_m |v_m^{n+1}| \leq \max_m |v_m^n|$$

if $bk/h^2 < 1/2$ & $h < 2b/a$ and $2bk/h^2 + ak/h \leq 1$ respectively. Discuss advantage and disadvantage of scheme (6.4.7) by comparing scheme (6.4.7) and scheme (6.4.2).

4′ Show that the scheme

$$\begin{aligned}
&\frac{u_m^{k+1}-u_m^k}{k} \\
= \quad & \frac{a}{2}\left(\frac{u_{m+1}^{k+1}-2u_m^{k+1}+u_{m-1}^{k+1}}{h^2}+\frac{u_{m+1}^{k}-2u_m^{k}+u_{m-1}^{k}}{h^2}\right) \\
& +\frac{b}{2}\left(\frac{u_{m+1}^{k+1}-u_{m-1}^{k+1}}{2h}+\frac{u_{m+1}^{k}-u_{m-1}^{k}}{2h}\right)
\end{aligned}$$

is unconditionally stable.

Section 7.2

1′ Consider the following parabolic partial differential equation:

$$\frac{\partial u}{\partial \tau} = a_{11}\frac{\partial^2 u}{\partial x^2} + 2a_{12}\frac{\partial^2 u}{\partial x \partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} + b_1\frac{\partial u}{\partial x} + b_2\frac{\partial u}{\partial y},$$

where $a_{11}(x, y, \tau) \geq 0, a_{22}(x, y, \tau) \geq 0, a_{12}(x, y, \tau) = \rho_{12}(x, y, \tau)\sqrt{a_{11}a_{12}}$ with $\rho_{12} \in [-1, 1]$, and b_1, b_2 are any functions of x, y, τ. This equation can be approximated by

i)

$$\begin{aligned}
&\frac{u_{m,n}^{k+1} - u_{m,n}^{k}}{\Delta\tau} \\
= \;& \frac{a_{11,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{u_{m+1,n}^{k+1} - 2u_{m,n}^{k+1} + u_{m-1,n}^{k+1}}{\Delta x^2} + \frac{u_{m+1,n}^{k} - 2u_{m,n}^{k} + u_{m-1,n}^{k}}{\Delta x^2}\right) \\
&+a_{12,m,n}^{k+\frac{1}{2}}\left(\frac{u_{m+1,n+1}^{k+1} - u_{m+1,n-1}^{k+1} - u_{m-1,n+1}^{k+1} + u_{m-1,n-1}^{k+1}}{4\Delta x\Delta y}\right. \\
&\qquad\left.+\frac{u_{m+1,n+1}^{k} - u_{m+1,n-1}^{k} - u_{m-1,n+1}^{k} + u_{m-1,n-1}^{k}}{4\Delta x\Delta y}\right) \\
&+\frac{a_{22,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{u_{m,n+1}^{k+1} - 2u_{m,n}^{k+1} + u_{m,n-1}^{k+1}}{\Delta y^2}\right. \\
&\qquad\left.+\frac{u_{m,n+1}^{k} - 2u_{m,n}^{k} + u_{m,n-1}^{k}}{\Delta y^2}\right) \\
&+\frac{b_{1,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{u_{m+1,n}^{k+1} - u_{m-1,n}^{k+1}}{2\Delta x} + \frac{u_{m+1,n}^{k} - u_{m-1,n}^{k}}{2\Delta x}\right) \\
&+\frac{b_{2,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{u_{m,n+1}^{k+1} - u_{m,n-1}^{k+1}}{2\Delta y} + \frac{u_{m,n+1}^{k} - u_{m,n-1}^{k}}{2\Delta y}\right)
\end{aligned}$$

or

ii)

$$
\begin{aligned}
&\frac{u^{k+1}_{m,n}-u^{k}_{m,n}}{\Delta\tau}\\
=\ &\frac{a^{k+\frac{1}{2}}_{11,m,n}}{2}\left(\frac{u^{k+1}_{m+1,n}-2u^{k+1}_{m,n}+u^{k+1}_{m-1,n}}{\Delta x^2}+\frac{u^{k}_{m+1,n}-2u^{k}_{m,n}+u^{k}_{m-1,n}}{\Delta x^2}\right)\\
&+a^{k+\frac{1}{2}}_{12,m,n}\left(\frac{u^{k+1}_{m+1,n+1}-u^{k+1}_{m+1,n-1}-u^{k+1}_{m-1,n+1}+u^{k+1}_{m-1,n-1}}{4\Delta x\Delta y}\right.\\
&\qquad\left.+\frac{u^{k}_{m+1,n+1}-u^{k}_{m+1,n-1}-u^{k}_{m-1,n+1}+u^{k}_{m-1,n-1}}{4\Delta x\Delta y}\right)\\
&+\frac{a^{k+\frac{1}{2}}_{22,m,n}}{2}\left(\frac{u^{k+1}_{m,n+1}-2u^{k+1}_{m,n}+u^{k+1}_{m,n-1}}{\Delta y^2}\right.\\
&\qquad\left.+\frac{u^{k}_{m,n+1}-2u^{k}_{m,n}+u^{k}_{m,n-1}}{\Delta y^2}\right)\\
&+\frac{b^{k+\frac{1}{2}}_{1,m,n}}{2}\left(\frac{-u^{k+1}_{m+2,n}+4u^{k+1}_{m+1,n}-3u^{k+1}_{m,n}}{2\Delta x}\right.\\
&\qquad\left.+\frac{-u^{k}_{m+2,n}+4u^{k}_{m+1,n}-3u^{k}_{m,n}}{2\Delta x}\right)\\
&+\frac{b^{k+\frac{1}{2}}_{2,m,n}}{2}\left(\frac{3u^{k+1}_{m,n}-4u^{k+1}_{m,n-1}+u^{k+1}_{m,n-2}}{2\Delta y}\right.\\
&\qquad\left.+\frac{3u^{k}_{m,n}-4u^{k}_{m,n-1}+u^{k}_{m,n-2}}{2\Delta y}\right)
\end{aligned}
$$

if $b_1(x,y,\tau)\ge 0$ and $b_2(x,y,\tau)\le 0$. By the von Neumann method, show that they are stable.

(Hint:

(a) First show that the amplification factor g can be written as $g=\frac{1+a+ib}{1-a-ib}$.

(b) Then show that $|g|^2\le 1$ is equivalent to $|1-a-ib|^2-|1+a+ib|^2=-4a\ge 0$.

(c) Finally show $-4a\ge 0$ by using the following inequalities: i). $A^2+B^2+2\rho AB=(A+\rho B)^2+B^2(1-\rho^2)\ge 0$ if $|\rho|\le 1$; ii). $\cos 2\theta-4\cos\theta+3=2(\cos\theta-1)^2\ge 0$.

)

Section 11.2

1′ Consider the leapfrog scheme

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0, \quad m = 1, 2, \cdots,$$

with one of the following schemes at boundary

(a) $v_0^{n+1} = v_1^{n+1} + \beta^{n+1}$,
(b) $v_0^{n+1} = 2v_1^{n+1} - v_2^{n+1} + \beta^{n+1}$,
(c) $v_0^{n+1} = v_1^n + \beta^{n+1}$,
(d) $v_0^{n+1} = 2v_1^n - v_2^{n-1} + \beta^{n+1}$.

Using the Laplace transform given in the book, show that for case (a) or (b), the method is unstable and for case (c) or (d), the method is stable for $ak/h \in (-1, 0]$,

2′ Suppose we have proved that the leapfrog scheme is stable for initial-value problem if $-1 < ak/h < 1$. Consider the following schemes at boundary

(a) $v_0^{n+1} = v_1^{n+1}$,
(b) $v_0^{n+1} = 2v_1^{n+1} - v_2^{n+1}$,
(c) $v_0^{n+1} = v_1^n$,
(d) $v_0^{n+1} = 2v_1^n - v_2^{n-1}$.

By theorem 11.3.3, show that the leapfrog scheme

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0, \quad m = 1, 2, \cdots,$$

with the boundary scheme (a) or (b) is unstable for $ak/h \in (-1, 0]$, with the boundary scheme (c) or (d) is stable for $ak/h \in (-1, 0]$, and with the boundary scheme (a), (b), (c) or (d) is unstable for $ak/h \in [0, 1)$. Also explain why all the computation for $ak/h \in [0, 1)$ is unstable and how this problem should be fixed.

Section 11.3

0′ (THIS SHOULD REPLACE 2’ IN SECTION 11.2) Suppose we have proved that the leapfrog scheme is stable for initial-value problem if $-1 < ak/h < 1$. Consider the following schemes at boundary

(a) $v_0^{n+1} = v_1^{n+1}$,

(b) $v_0^{n+1} = 2v_1^{n+1} - v_2^{n+1}$,

(c) $v_0^{n+1} = v_1^n$,

(d) $v_0^{n+1} = 2v_1^n - v_2^{n-1}$.

By theorem 11.3.3, show that the leapfrog scheme

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0, \quad m = 1, 2, \cdots,$$

with the boundary scheme (a) or (b) is unstable for $ak/h \in (-1, 0]$, with the boundary scheme (c) or (d) is stable for $ak/h \in (-1, 0]$, and with the boundary scheme (a), (b), (c) or (d) is unstable for $ak/h \in [0, 1)$. Also explain why all the computation for $ak/h \in [0, 1)$ is unstable and how this problem should be fixed.

1′ Suppose we have proved that the Crank–Nicolson scheme is unconditionally stable for initial-value problem. Consider the following schemes at boundary

(a) $v_0^{n+1} = v_1^{n+1}$,

(b) $v_0^{n+1} = 2v_1^{n+1} - v_2^{n+1}$,

(c) $v_0^{n+1} = v_1^n$,

(d) $v_0^{n+1} = 2v_1^n - v_2^{n-1}$.

By theorem 11.3.3, show that the Crank–Nicolson scheme

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1} + v_{m+1}^n - v_{m-1}^n}{4h} = 0, \quad m = 1, 2, \cdots,$$

with the boundary scheme (a) or (b) is stable for $ak/h \in (-\infty, 0]$. with the boundary scheme (c) or (d) is stable for $ak/h \in (-2, 0]$ and unstable for $ak/h \in (-\infty, -2]$, and with the boundary scheme (a), (b), (c), or (d) is unstable for $ak/h \in (0, \infty)$. Also explain why all the computation for $ak/h \in [0, \infty)$ is unstable and how this problem should be fixed.

1″ Suppose we have proved that the Lax–Wendroff scheme is stable for initial-value problem if $-1 \le ak/h \le 1$. Consider the following schemes at boundary

(a) $v_0^{n+1} = v_1^{n+1}$,

(b) $v_0^{n+1} = 2v_1^{n+1} - v_2^{n+1}$,

(c) $v_0^{n+1} = v_1^n$,

(d) $v_0^{n+1} = 2v_1^n - v_2^{n-1}$.

By theorem 11.3.3, show that the Lax–Wendroff scheme

$$v_m^{n+1} = v_m^n - \frac{ak}{2h}\left(v_{m+1}^n - v_{m-1}^n\right) + \frac{a^2k^2}{2h^2}\left(v_{m+1}^n - 2v_m^n + v_{m-1}^n\right), \quad m = 1, 2, \cdots,$$

with the boundary scheme (a), (b), (c) or (d) is stable for $ak/h \in [-1, 0]$, and with the boundary scheme (a), (b), (c), or (d) is unstable for $ak/h \in (0, 1]$. Also explain why all the computation for $ak/h \in (0, 1]$ is unstable and how this problem should be fixed.

2′ Suppose we have proved that the Lax–Wendroff scheme is stable for initial-value problem if $-1 \le ak/h \le 1$. By theorem 11.3.3, show that the Lax–Wendroff scheme

$$v_m^{n+1} = v_m^n - \frac{ak}{2h}\left(v_{m+1}^n - v_{m-1}^n\right) + \frac{a^2k^2}{2h^2}\left(v_{m+1}^n - 2v_m^n + v_{m-1}^n\right), \quad m = 1, 2, \cdots,$$

with the boundary scheme

$$v_0^{n+1} = v_0^n - \frac{ak}{h}\left(v_1^n - v_0^n\right),$$

is stable if $-1 \le ak/h \le 0$. (Hint: First show that the solution of the system

$$\begin{cases} z = 1 - \frac{ak}{2h}\left(K - K^{-1}\right) + \frac{a^2k^2}{2h^2}\left(K - 2 + K^{-1}\right), \\ z = 1 - \frac{ak}{h}\left(K - 1\right) \end{cases}$$

is $z = K = 1$ if $ak/h \ne 0$ and $ak/h \ne -1$; then show (i) $K_+(1) = 1$ for $a < 0$, (ii) when $ak/h = 0$, the method is stable, and (iii) when $ak/h = -1$, $K = z$ and $K_+\left(e^{i\theta}\right) = e^{i\theta}$.)

3′ Suppose we have proved that the Crank–Nicolson scheme is unconditionally stable for initial-value problem. By theorem 11.3.3, show that the Crank–Nicolson scheme

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1} + v_{m+1}^n - v_{m-1}^n}{4h} = 0, \quad m = 1, 2, \cdots,$$

with the boundary scheme

$$\frac{v_0^{n+1} - v_0^n + v_1^{n+1} - v_1^n}{2k} + a\frac{v_1^{n+1} - v_0^{n+1} + v_1^n - v_0^n}{2h} = 0,$$

is stable if $ak/h \le 0$. (Hint: First show that the solution of the system

$$\begin{cases} \frac{z-1}{k} + a\frac{z(K-K^{-1})+K-K^{-1}}{4h} = 0, \\ \frac{z-1+(z-1)K}{2k} + a\frac{z(K-1)+K-1}{2h} = 0 \end{cases}$$

is $z = K = 1$ if $ak/h \ne 0$; then show (i) $K_+(1) = 1$ for $a < 0$, and (ii) when $ak/h = 0$, the method is stable.)

4′ Suppose we have proved that the Lax–Wendroff scheme is stable for initial-value problem if $-1 \le ak/h \le 1$. By theorem 11.3.3, show that the Lax–Wendroff scheme

$$v_m^{n+1} = v_m^n - \frac{ak}{2h}\left(v_{m+1}^n - v_{m-1}^n\right) + \frac{a^2k^2}{2h^2}\left(v_{m+1}^n - 2v_m^n + v_{m-1}^n\right), \quad m = 1, 2, \cdots,$$

with the boundary scheme

$$v_0^{n+1} = v_0^n - \frac{ak}{2h}\left(-v_2^n + 4v_1^n - 3v_0^n\right) + \frac{a^2k^2}{2h^2}\left(v_2^n - 2v_1^n + v_0^n\right),$$

is stable if $-1 \le ak/h \le 0$. (Hint: First show that the solution of the system

$$\begin{cases} z = 1 - \frac{ak}{2h}\left(K - K^{-1}\right) + \frac{a^2k^2}{2h^2}\left(K - 2 + K^{-1}\right), \\ z = 1 - \frac{ak}{2h}\left(-K^2 + 4K - 3\right) + \frac{a^2k^2}{2h^2}\left(K^2 - 2K + 1\right) \end{cases}$$

is $z = K = 1$ if $ak/h \ne 0$ and $ak/h \ne -1$; then show (i) $K_+(1) = 1$ for $a < 0$, (ii) when $ak/h = 0$, the method is stable, and (iii) when $ak/h = -1$, $K = z$ and $K_+\left(e^{i\theta}\right) = e^{i\theta}$.)

5′ Suppose we have proved that the Crank–Nicolson scheme is unconditionally stable for initial-value problem. By theorem 11.3.3, show that the Crank–Nicolson scheme

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1} + v_{m+1}^n - v_{m-1}^n}{4h} = 0, \quad m = 1, 2, \cdots,$$

with the boundary scheme

$$\frac{v_0^{n+1} - v_0^n}{k} + a\frac{-v_2^{n+1} + 4v_1^{n+1} - 3v_0^{n+1} - v_2^n + 4v_1^n - 3v_0^n}{4h} = 0,$$

is stable if $ak/h \le 0$. (Hint: First show that the solution of the system

$$\begin{cases} \frac{z-1}{k} + a\frac{z(K-K^{-1})+K-K^{-1}}{4h} = 0, \\ \frac{z-1}{k} + a\frac{z(-K^2+4K-3)-K^2+4K-3}{4h} = 0 \end{cases}$$

is $z = K = 1$ if $ak/h \ne 0$; then show (i) $K_+(1) = 1$ for $a < 0$, and (ii) when $ak/h = 0$, the method is stable.)

$6'$ Suppose we have proved that the leapfrog scheme is stable for initial-value problem if $-1 < ak/h < 1$. By theorem 11.3.3, show that the leapfrog scheme

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0, \quad m = 1, 2, \cdots,$$

with the boundary scheme

$$\frac{v_0^{n+1} - v_0^n + v_1^{n+1} - v_1^n}{2k} + a\frac{v_1^{n+1} - v_0^{n+1} + v_1^n - v_0^n}{2h} = 0,$$

is stable if $-1 < ak/h \leq 0$. (Hint: First show that the solution of the system

$$\begin{cases} \frac{z - z^{-1}}{2k} + a\frac{K - K^{-1}}{2h} = 0, \\ \frac{z - 1 + (z-1)K}{2k} + a\frac{z(K-1) + K - 1}{2h} = 0 \end{cases}$$

is $K = z$ and $K = 1/z$ if $ak/h \neq 0$; then show (i) only $z = K = 1$ and $z = K = -1$ need to be considered, and $K_+(1) = 1$ and $K_+(-1) = -1$ for $a < 0$, and (ii) when $ak/h = 0$, the method is stable.)

$7'$ Suppose we have proved that the Crank–Nicolson scheme is unconditionally stable for initial-value problem. By theorem 11.3.3, show that the Crank–Nicolson scheme

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1} + v_{m+1}^n - v_{m-1}^n}{4h} = 0, \quad m = 1, 2, \cdots,$$

with the boundary scheme

$$v_0^{n+1} \;=\; v_0^n - \frac{ak}{h}\left(v_1^n - v_0^n\right),$$

is stable if $-2 < ak/h \leq 0$, and is unstable if $ak/h \leq -2$. (Hint: First show that the solution of the system

$$\begin{cases} \frac{z-1}{k} + a\frac{z(K - K^{-1}) + K - K^{-1}}{4h} = 0, \\ z = 1 - \frac{ak}{h}(K - 1) \end{cases}$$

is $z = K = 1$ and $K = \dfrac{2}{-ak/h} - 1$ if $ak/h \neq 0$; then show (i) $K_+(1) = 1$ for $a < 0$, (ii) when $-2 < ak/h < 0$, the methed is stable and when $ak/h \leq -2$, the methed is unstable, and (iii) when $ak/h = 0$, the methed is stable.)