ON STABILITY AND CONVERGENCE OF DIFFERENCE SCHEMES
FOR QUASILINEAR HYPERBOLIC INITIAL-BOUNDARY-VALUE PROBLEMS

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Abstract

In the seventies[2,3] we obtained some results on stability of difference schemes for initial-boundary-value problems of linear diagonalized hyperbolic systems in two independent variables. Later[4] these results were extended to general linear hyperbolic systems with "moving boundaries" and some convergence theorems were established. In [1], we completed some proofs of 'global' convergence of difference schemes for general quasilinear hyperbolic initial-boundary-value problems with moving boundaries. Recently, more results on convergence have been derived. From these results we know that when we solve a quasilinear hyperbolic system using certain second order Singularity-Separating difference methods[3] (separating discontinuities, weak discontinuities etc.), the approximate solution will converge to the exact solution with a convergence rate of Δt^2 in L_2 norm, no matter whether or not there exist some discontinuities, such as shocks, contact discontinuities. In this paper we shall summarize our main results on this subject.

1. INITIAL-BOUNDARY-VALUE PROBLEMS

Let us consider the following initial-boundary-value problem for quasilinear hyperbolic systems in two independent variables.

1. A quasilinear hyperbolic system

$$\frac{\partial \bar{U}}{\partial t} + \bar{A}(\bar{U}, x, t)\frac{\partial \bar{U}}{\partial x} = \bar{F}(\bar{U}, x, t) \tag{1.1}$$

is given in L regions: $x_{\ell-1}(t) \le x \le x_\ell(t),\ 0 \le t \le T$, $\ell = 1,2,\cdots,L$.

2. On external boundaries and internal boundaries $x = x_\ell(t)$, $\ell = 0,1,\cdots,L$, a number of nonlinear boundary conditions are prescribed:

$$\begin{cases} B_0(\bar{U}_0^+, x_0, z_0, t) = 0, \\ B_\ell(\bar{U}_\ell^-, \bar{U}_\ell^+, x_\ell, z_\ell, t) = 0, \quad \ell = 1,2,\cdots, L-1, \\ B_L(\bar{U}_L^-, x_L, z_L, t) = 0, \end{cases} \tag{1.2}$$

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where

$$z_\ell = \frac{dx_\ell(t)}{dt}, \qquad \ell = 0,1,\cdots,L. \tag{1.3}$$

3. At t = 0 initial values are specified:

$$\begin{cases} \bar{U}(x,0) = \bar{D}_\ell(x), \text{ for } x_{\ell-1}(0) \le x \le x_\ell(0), \quad \ell = 1,2,\cdots,L, \\ x_\ell(0) = c_{0,\ell},\ z_\ell(0) = c_{1,\ell}, \quad \ell = 0,1,\cdots,L. \end{cases} \tag{1.4}$$

It is required to determine $\bar{U}$ in the L regions and $x_\ell(t)$, $z_\ell(t)$ for $0 \le t \le T$, $\ell = 0,1,\cdots,L$. Here $\bar{U},\bar{F},\bar{D}_\ell$ are N_1 - dimensional vectors, $\bar{A}$— an $N_1 \times N_1$ matrix, B_ℓ— ν_ℓ - dimensional vectors and $\bar{U}_\ell^\pm \equiv \bar{U}_\ell^\pm(t) = \lim_{x\to x_\ell(t)\pm 0} \bar{U}(x,t)$. Without loss of generality, we assume that the initial values (1.4) satisfy the boundary conditions (1.2). In fact, if initial values are discontinuous at some point, we should solve a Riemann problem to determine how many and what types of internal boundaries need to be introduced; if initial values do not satisfy some external boundary conditions, a similar procedure should be done. Therefore the initial values always match the boundary conditions. The hyperbolic type means that $\bar{A}$ has the following expression

$$\bar{A} = \bar{G}^{-1}\ \bar{\Lambda}\ \bar{G}, \tag{1.5}$$

where

$$\bar{G} = \begin{bmatrix} \bar{G}_1^* \\ \bar{G}_2^* \\ \vdots \\ \bar{G}_{N_1}^* \end{bmatrix} \quad \text{and} \quad \bar{\Lambda} = \begin{bmatrix} \bar{\lambda}_1 & & & 0 \\ & \bar{\lambda}_2 & & \\ & & \ddots & \\ 0 & & & \bar{\lambda}_{N_1} \end{bmatrix}$$

are a nonsingular real matrix and a real diagonal matrix respectively, $\bar{G}_n^*$ being the transpose of a column vector $\bar{G}_n$. In the following we assume $\bar{\lambda}_1 \ge \bar{\lambda}_2 \ge \cdots \ge \bar{\lambda}_{N_1}$ and define $p_\ell \equiv$ the number of $\bar{\lambda}_{n,\ell}^+$ which are less than or equal to $\frac{dx_\ell}{dt}$, and $q_\ell \equiv$ the number of $\bar{\lambda}_{n,\ell}^-$ which are greater than or equal to $\frac{dx_\ell}{dt}$, where $\bar{\lambda}_{n,\ell}^\pm = \lim_{x=x_\ell(t)\pm 0} \bar{\lambda}_n(x,t)$. We require the boundary conditions (1.2) are compatible with the equation (1.1). That is, we suppose that ν_ℓ satisfy

$$\begin{cases} \nu_0 + p_0 = N_1 + 1, \\ \nu_\ell + p_\ell + q_\ell = 2N_1 + 1, \quad \ell = 1,2,\cdots,L-1, \\ \nu_L + q_L = N_1 + 1, \end{cases} \tag{1.6}$$

and furthermore, the systems

$$\begin{cases} B_0(\bar{U}_0^+, x_0, z_0, t) = 0, \\ \bar{G}_{(0)}^+\ \bar{U}_0^+ = F_0^+, \end{cases}$$

$$\begin{cases} \bar{G}^-_{(\ell)}\ \bar{U}^-_\ell = F^-_\ell , \\ B_\ell(\bar{U}^-_\ell,\ \bar{U}^+_\ell,\ x_\ell,\ z_\ell,\ t) = 0, \qquad \ell = 1,2,\cdots, L-1, \\ \bar{G}^+_{(\ell)}\ \bar{U}^+_\ell = F^+ , \end{cases} \tag{1.7}$$

$$\begin{cases} \bar{G}^-_{(L)}\ \bar{U}^-_L = F^-_L , \\ B_L(\bar{U}^-_L,\ x_L,\ z_L,\ t) = 0 \end{cases}$$

always have "entropy-satisfying" solutions $\bar{U}^+_0$, z_0; $\bar{U}^-_\ell$, $\bar{U}^+_\ell$, z_ℓ, $\ell=1,2,\cdots,L-1$; $\bar{U}^-_L$, z_L, respectively if reasonable x_ℓ, F^+_ℓ, F^-_ℓ and t are given. Here F^+_ℓ, F^-_ℓ are p_ℓ-, q_ℓ - dimensional vectors and

$$\bar{G}^+_{(\ell)} = \begin{bmatrix} \bar{G}^*_{N_1-p_\ell+1} \\ \vdots \\ \bar{G}^*_{N_1} \end{bmatrix}\Bigg|_{x=x_\ell(t)+0}, \qquad \bar{G}^-_{(\ell)} = \begin{bmatrix} \bar{G}^*_1 \\ \vdots \\ \bar{G}^*_{q_\ell} \end{bmatrix}\Bigg|_{x=x_\ell(t)-0}.$$

In order to make numerical computation easy, we introduce the following coordinate transformation

$$\begin{cases} \xi = \dfrac{x - x_{\ell-1}(t)}{x_\ell(t)-x_{\ell-1}(t)} + \ell - 1, \quad \text{if } x_{\ell-1}(t) \le x \le x_\ell(t), \\ \qquad\qquad \ell = 1,2,\cdots, L, \\ t = t. \end{cases}$$

Clearly, the boundary $x = x_\ell(t)$ in the x - t coordinate system corresponds to the straight line $\xi = \ell$ in the ξ - t coordinate system. For $\bar{U}$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \frac{\partial \xi}{\partial t}\ \frac{\partial}{\partial \xi} ,$$

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\ \frac{\partial}{\partial \xi},$$

where

$$\begin{cases} \dfrac{\partial \xi}{\partial t} = \dfrac{(-z_{\ell-1})(x_\ell - x_{\ell-1}) - (x - x_{\ell-1})(z_\ell - z_{\ell-1})}{(x_\ell - x_{\ell-1})^2} , \\ \dfrac{\partial \xi}{\partial x} = \dfrac{1}{x_\ell - x_{\ell-1}} , \text{ for } x_{\ell-1} \le x \le x_\ell,\ \ell = 1,2,\cdots,L. \end{cases}$$

Therefore (1.1) can be transformed into

$$\frac{\partial \bar{U}}{\partial t} + \bar{A}_1\ \frac{\partial \bar{U}}{\partial \xi} = \bar{F}, \tag{1.8}$$

where

$$\bar{A}_1(\bar{U},X,Z,\xi,t) = \frac{\partial \xi}{\partial t}\ I_{N_1} + \frac{\partial \xi}{\partial x}\ \bar{A} ,$$

I_{N_1} being an $N_1 \times N_1$ unit matrix, $X = (x_0, x_1, \cdots, x_L)^T$, and $Z = (z_0,\ z_1, \cdots,\ z_L)^T$.

For convenience to theoretical proof, in what follows, the ordinary differential relations (1.3) are understood as some hyperbolic partial differential equations. Therefore, defining

$$U = \begin{bmatrix} \bar{U} \\ X \end{bmatrix} , \qquad A = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & 0 \end{bmatrix} , \qquad F = \begin{bmatrix} \bar{F} \\ Z \end{bmatrix} ,$$

we can combine (1.8) and (1.3) to

$$\frac{\partial U}{\partial t} + A(U,Z,\xi,t)\frac{\partial U}{\partial \xi} = F(U,Z,\xi,t).$$

Thus, (1.1)-(1.4) can be rewritten as:

1. A quasilinear hyperbolic system

$$\frac{\partial U}{\partial t} + A(U,Z,\xi,t)\ \frac{\partial U}{\partial \xi} = F(U,Z,\xi,t) \tag{1.9}$$

   is given in L regions: $\ell-1 \le \xi \le \ell$, $0 \le t \le T$, $\ell = 1,2,\cdots, L$.

2. On the straight boundaries $\xi = \ell$, $\ell = 0,1,\cdots,L$, a number of nonlinear boundary conditions are prescribed:

$$\begin{cases} B_0(U^+_0,\ z_0,\ t) = 0, \\ B_\ell(U^-_\ell,\ U^+_\ell,\ z_\ell,\ t) = 0, \quad \ell = 1,2,\cdots,L-1, \\ B_L(U^-_L,\ z_L,\ t) = 0. \end{cases} \tag{1.10}$$

3. At t = 0, initial values are specified:

$$\begin{cases} U(\xi,0) = D_\ell(\xi) \quad \text{for} \quad \ell-1 \le \xi \le \ell, \quad \ell = 1,2,\cdots, L, \\ Z(0) = C_1 , \end{cases} \tag{1.11}$$

where D_ℓ and C_1 are $(N_1 + L + 1)$- and $(L + 1)$-dimensional vectors respectively. We need to determine U in the L regions and Z for $0 \le t \le T$. Let

$$G = \begin{bmatrix} \bar{G} & 0 \\ 0 & I_{L+1} \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \frac{\partial \xi}{\partial t} I_{N_1} + \frac{\partial \xi}{\partial x} \bar{\Lambda} & 0 \\ 0 & 0 \end{bmatrix},$$

we have

$$A = G^{-1} \Lambda G.$$

Therefore, (1.9) can be written as

$$G \frac{\partial U}{\partial t} + \Lambda G \frac{\partial U}{\partial \xi} = GF. \qquad (1.12)$$

2. DIFFERENCE SCHEMES

The system (1.12) can further be rewritten in the form

$$G_n^* \frac{\partial U}{\partial t} + \lambda_n G_n^* \frac{\partial U}{\partial \xi} = f_n, \quad n = 1,2,\cdots,N, \qquad (2.1)$$

where G_n^* is the n-th row of G, λ_n is the n-th diagonal element of Λ, f_n is the n-th component of GF and $N = N_1 + L + 1$. In what follows, we will discuss the discretization of (2.1).

In each subregion $\ell-1 \le \xi \le \ell$, $0 \le t \le T$, we make a rectangular mesh with mesh sizes $\Delta\xi = 1/M_\ell$ and Δt, M_ℓ being an integer. For convenience, in what follows, we assume $M = M_\ell$ for all ℓ, $f^k_{\ell,m}$ denotes the value of f at $\xi = \ell - 1 + m\Delta\xi$, $t = k\Delta t$ and we define $\sigma = \lambda\Delta t/\Delta\xi$. Moreover, we assume $\frac{\Delta t}{\Delta \xi}$ is bounded. In the ℓ-th subregion for each $\lambda^k_{n,\ell}$ we define a set $\underline{g}(\lambda^k_{n,\ell})$ as follows:

$$\underline{g}(\lambda^k_{n,\ell}) = \begin{cases} \{1,2,\cdots,M\}, & \text{if } \lambda^k_{n,\ell,0} > 0 \text{ and } \lambda^k_{n,\ell,M} \ge 0, \\ \{0,1,\cdots,M-1\}, & \text{if } \lambda^k_{n,\ell,0} \le 0 \text{ and } \lambda^k_{n,\ell,M} < 0, \\ \{0,1,\cdots,M\}, & \text{if } \lambda^k_{n,\ell,0} \le 0 \text{ and } \lambda^k_{n,\ell,M} \ge 0, \\ \{1,2,\cdots,M-1\}, & \text{if } \lambda^k_{n,\ell,0} > 0 \text{ and } \lambda^k_{n,\ell,M} < 0. \end{cases} \qquad (2.2)$$

We suppose that when n, ℓ is fixed, $\underline{g}(\lambda^k_{n,\ell})$ is the same for all k. Therefore, from now on, we use the symbol $\underline{g}(\lambda_{n,\ell})$ instead of $\underline{g}(\lambda^k_{n,\ell})$. For each $\lambda_{n,\ell}$ (2.1) is approximated by a second order accurate system of difference equations of the following form

$$\sum_{h=h_1(m)}^{h_2(m)} R^{k+\frac12}_{h,n,\ell,m} U^{k+1}_{\ell,m+h} = \sum_{h=h_1(m)}^{h_2(m)} S^{k+\frac12}_{h,n,\ell,m} U^k_{\ell,m+h} + \Delta t f^{k+\frac12}_{n,\ell,m}, \qquad (2.3)$$

$$m \in \underline{g}(\lambda_{n,\ell}),$$

where for any m hold the relations

$$0 \le m+h_1(m) \le m + h_2(m) \le M, \quad \max_m \{|h_1(m)|, \; |h_2(m)|\} \le H,$$

H being a positive integer,

and $R^{k+\frac12}_{h,n,\ell,m}$, $S^{k+\frac12}_{h,n,\ell,m}$ and $f^{k+\frac12}_{n,\ell,m}$ depend on $U^{k+\frac12}_{\ell,m+j}$, $Z^{k+\frac12}$, $U^k_{\ell,m+j}$, Z^k besides ξ and t, j satisfying $h_1(m) \le j \le h_2(m)$. According to the consistency, among $R^{k+\frac12}_{h,n,\ell,m}$, $S^{k+\frac12}_{h,n,\ell,m}$ exists the relation

$$\sum_{h=h_1(m)}^{h_2(m)} R^{k+\frac12}_{h,n,\ell,m} = \sum_{h=h_1(m)}^{h_2(m)} S^{k+\frac12}_{h,n,\ell,m} = 0(1). \qquad (2.4)$$

If $\lambda_{n,\ell,m} > 0$ (or < 0) for all m, using the implicit second order scheme for (2.1)

$$\frac12 \mu G^{*k+\frac12}_{m\mp\frac12} (U^{k+1}_m + U^{k+1}_{m\mp1}) + \frac12 \mu\sigma^{k+\frac12}_{m\mp\frac12} \mu G^{*k+\frac12}_{m\mp\frac12} \Delta_\mp U^{k+1}_m$$
$$= \frac12 \mu G^{*k+\frac12}_{m\mp\frac12} (U^k_m + U^k_{m\mp1}) - \frac12 \mu\sigma^{k+\frac12}_{m\mp\frac12} \mu G^{*k+\frac12}_{m\mp\frac12} \Delta_\mp U^k_m + \Delta t \, \mu f^{k+\frac12}_{m\mp\frac12} \qquad (2.5)$$

we can have a system of difference equations in the form (2.3). In (2.5) the subscripts n and ℓ are omitted, and we use the notation

$$\Delta_+ U_m = U_{m+1} - U_m, \quad \Delta_- U_m = U_m - U_{m-1},$$

$$\mu f_{m\mp\frac12} = \frac12 (f_m + f_{m\mp1}).$$

And the minus sign of $\mp$ should be adopted if $\lambda > 0$ and the plus sign if $\lambda < 0$.

If $\lambda_{n,\ell,0} > 0$ and $\lambda_{n,\ell,M} < 0$, using the explicit second order scheme for (2.1)

$$G^{*k+\frac12}_{m'} U^{k+1}_m = G^{*k+\frac12}_{m'} \Big(\frac12(1+\sigma^{k+\frac12}_{m'}) \sigma^{k+\frac12}_{m'} U^k_{m-1} + (1+\sigma^{k+\frac12}_{m'})(1-\sigma^{k+\frac12}_{m'}) U^k_m$$
$$- \frac12 (1-\sigma^{k+\frac12}_{m'}) \sigma^{k+\frac12}_{m'} U^k_{m+1}\Big) + \Delta t f^{k+\frac12}_{m'}, \qquad (2.6)$$

we can obtain a system of difference equations in form (2.3); if $\lambda_{n,\ell,0} \le 0$ and $\lambda_{n,\ell,M} \ge 0$, using the one-sided second order explicit scheme

$$G^{*k+\frac12}_{m'} U^{k+1}_m = G^{*k+\frac12}_{m'} \Big(-\frac12 (1-|\sigma^{k+\frac12}_{m'}|) |\sigma^{k+\frac12}_{m'}| U^k_{m\mp2}$$
$$+ |\sigma^{k+\frac12}_{m'}| (2-|\sigma^{k+\frac12}_{m'}|) U^k_{m\mp1} + \frac12 (2-|\sigma^{k+\frac12}_{m'}|)(1-|\sigma^{k+\frac12}_{m'}|) U^k_m\Big)$$
$$+ \Delta t f^{k+\frac12}_{m'}, \qquad (2.7)$$

a system in form (2.3) can also be obtained. In (2.6) and (2.7) the subscripts n and ℓ are also omitted and in (2.7) the minus sign of $\mp$ should be chosen if $\lambda \ge 0$ and the plus sign if $\lambda \le 0$. (Noticing that (2.7) becomes $G^{*k+\frac12}_{m'} U^{k+1}_m = G^{*k+\frac12}_{m'} U^k_m + \Delta t f^{k+\frac12}_{m'}$ if $\lambda = 0$, no matter which sign you choose, we know that it is no problem for the sign of equality to appear in both cases.) In order to guarantee a second order accuracy, $G^{*k+\frac12}_{m'}$ and $\sigma^{k+\frac12}_{m'}$ are computed by the linear interpolation formula

$$f_{m'} = (1 \mp (m - m'))f_m \mp (m' - m) f_{m\mp1},$$

where

$$m' = m - \frac{\lambda_m^{k+\frac{1}{2}} \frac{\Delta t}{2\Delta\xi}}{1 \mp (\lambda_{m\mp 1}^{k+\frac{1}{2}} - \lambda_m^{k+\frac{1}{2}}) \frac{\Delta t}{2\Delta\xi}} ,$$

and the minus sign of $\mp$ should be chosen if $\lambda \geq 0$ and the plus sign if $\lambda \leq 0$.

Consequently, second order accurate systems in form (2.3) are existent. We have given some schemes, which are in form (2.3) and can be applied to various cases, in [2] and [3]. For more details, please see [2] and [3].

In order to use scheme (2.3), we have to know $U^0_{\ell,m}$, Z^0, $U^{\frac{1}{2}}_{\ell,m}$ and $Z^{\frac{1}{2}}$ at the beginning. We use a first order scheme to get $U^{\frac{1}{2}}_{\ell,m}$ and $Z^{\frac{1}{2}}$ from $U^0_{\ell,m}$ and Z^0. If the superscripts of G and σ in (2.5)-(2.7) are changed to k from $k+\frac{1}{2}$, (2.5)-(2.7) can be applied to this purpose.

Therefore, the numerical procedure can be described as follows. At first, in order to get $U^{\frac{1}{2}}_{\ell,m}$ and $Z^{\frac{1}{2}}$ from $U^0_{\ell,m}$ and Z^0, the system

$$\begin{cases} \sum\limits_{h=h_1(m)}^{h_2(m)} R^0_{h,n,\ell,m} U^{\frac{1}{2}}_{\ell,m+h} = \sum\limits_{h=h_1(m)}^{h_2(m)} S^0_{h,n,\ell,m} U^0_{\ell,m+h} + \frac{1}{2}\Delta t\, f^0_{n,\ell,m} \\ \qquad \ell = 1,2,\cdots,L, \quad n=1,2,\cdots,N, \quad m \in \underline{g}(\lambda_{n,\ell}), \\ B_0(U^{\frac{1}{2}}_{1,0}, z^{\frac{1}{2}}_0, t^{\frac{1}{2}}) = 0, \\ B_\ell(U^{\frac{1}{2}}_{\ell,M}, U^{\frac{1}{2}}_{\ell+1,0}, z^{\frac{1}{2}}_\ell, t^{\frac{1}{2}}) = 0, \quad \ell = 1,2,\cdots, L-1, \\ B_L(U^{\frac{1}{2}}_{L,M}, z^{\frac{1}{2}}_L, t^{\frac{1}{2}}) = 0 \end{cases} \tag{2.8}$$

should be solved. Here $R^0_{h,n,\ell,m}$, $S^0_{h,n,\ell,m}$, $f^0_{n,\ell,m}$ depend only on $U^0_{\ell,m}$ and Z^0. Then for $k = 0, \frac{1}{2}, 1,\cdots, T/\Delta t - 1$, we solve

$$\begin{cases} \sum\limits_{h=h_1(m)}^{h_2(m)} R^{k+\frac{1}{2}}_{h,n,\ell,m} U^{k+1}_{\ell,m+h} = \sum\limits_{h=h_1(m)}^{h_2(m)} S^{k+\frac{1}{2}}_{h,n,\ell,m} U^k_{\ell,m+h} + \Delta t\, f^{k+\frac{1}{2}}_{n,\ell,m}, \\ \qquad \ell = 1,2,\cdots,L, \quad n = 1,2,\cdots,N, \; m \in \underline{g}(\lambda_{n,\ell}), \\ B_0(U^{k+1}_{1,0}, z^{k+1}_0, t^{k+1}) = 0, \\ B_\ell(U^{k+1}_{\ell,M}, U^{k+1}_{\ell+1,0}, z^{k+1}_\ell, t^{k+1}) = 0, \quad \ell = 1,2,\cdots,L-1, \\ B_L(U^{k+1}_{L,M}, z^{k+1}_L, t^{k+1}) = 0. \end{cases} \tag{2.9}$$

Let $\underline{U}$ be a vector whose components are $U_{1,0}, U_{1,1},\cdots,U_{1,M},\cdots,U_{L,0}, U_{L,1},\cdots,U_{L,M}$ from the top to the bottom and U_b be a vector whose components are $U_{1,0}, U_{1,M}, U_{2,0}, U_{2,M}, U_{3,0}, U_{3,M},\cdots,U_{L,0}, U_{L,M}$ from the top to the bottom. $\underline{R}$ denotes a matrix whose every row consists of the coefficients on the left hand side of a difference

great number of zeros. The rows of matrix $\underline{S}$ and the components of vector $\underline{F}$ may be defined equivalently. Then (2.8) and (2.9) can be rewritten as

$$\begin{cases} \underline{R}^0 \underline{U}^{\frac{1}{2}} = \underline{S}^0 \underline{U}^0 + \frac{1}{2}\Delta t\, \underline{F}^0, \\ B(U_b^{\frac{1}{2}}, Z^{\frac{1}{2}}, t^{\frac{1}{2}}) = 0, \end{cases} \tag{2.10}$$

$$\begin{cases} \underline{R}^{k+\frac{1}{2}} \underline{U}^{k+1} = \underline{S}^{k+\frac{1}{2}} \underline{U}^k + \Delta t\, \underline{F}^{k+\frac{1}{2}}, \\ B(U_b^{k+1}, Z^{k+1}, t^{k+1}) = 0, \\ \qquad k = 0, \frac{1}{2},\cdots, T/\Delta t - 1. \end{cases} \tag{2.11}$$

Here $B(U_b, Z, t) = 0$ are nonlinear equations which correspond to these nonlinear boundary conditions in (2.8) or (2.9). According to their definitions, $\underline{R}^0$, $\underline{S}^0$, $\underline{F}^0$ depend on $\underline{U}^0$, Z^0, t^0, and $\underline{R}^{k+\frac{1}{2}}$, $\underline{S}^{k+\frac{1}{2}}$, $\underline{F}^{k+\frac{1}{2}}$ depend on $\underline{U}^{k+\frac{1}{2}}$, $Z^{k+\frac{1}{2}}$, $t^{k+\frac{1}{2}}$, $\underline{U}^k, Z^k, t^k$, $k=0, \frac{1}{2}, 1,\cdots,T/\Delta t - 1$, i.e., $\underline{R}^0 \equiv \underline{R}(\underline{U}^0, Z^0, t),\cdots$, and $\underline{R}^{k+\frac{1}{2}} \equiv \underline{R}(\underline{U}^{k+\frac{1}{2}}, Z^{k+\frac{1}{2}}, t^{k+\frac{1}{2}}, \underline{U}^k, Z^k, t^k),\cdots$, $k= 0,\frac{1}{2},\cdots,T/\Delta t - 1$.

Noticing (2.4), we can rewrite (2.3) in the form

$$\sum_{h=h_1(m)}^{h_2(m)} R^{k+\frac{1}{2}}_{h,n,\ell,m}\, \delta U^k_{\ell,m+h} = \sum_{h=h_1(m)}^{h_2(m)-1} Q^{k+\frac{1}{2}}_{h,n,\ell,m}\, \Delta_+ U^k_{\ell,m+h} + \Delta t\, f^{k+\frac{1}{2}}_{n,\ell,m}, \tag{2.12}$$

where $\delta U^k_{\ell,m+h} = U^{k+1}_{\ell,m+h} - U^k_{\ell,m+h}$, $\Delta_+ U^k_{\ell,m+h} = U^k_{\ell,m+h+1} - U^k_{\ell,m+h}$, and

$$Q^{k+\frac{1}{2}}_{h,n,\ell,m} = \sum_{j=h_1(m)}^{h} (R^{k+\frac{1}{2}}_{j,n,\ell,m} - S^{k+\frac{1}{2}}_{j,n,\ell,m}).$$

Therefore (2.10) and (2.11) can also be written in the following form

$$\begin{cases} \underline{R}^0\, \delta_{\frac{1}{2}} \underline{U}^0 = \underline{Q}^0\, \Delta_+ \underline{U}^0 + \frac{1}{2}\Delta t\, \underline{F}^0, \\ B(U_b^{\frac{1}{2}}, Z^{\frac{1}{2}}, t^{\frac{1}{2}}) = 0, \end{cases} \tag{2.13}$$

and

$$\begin{cases} \underline{R}^{k+\frac{1}{2}}\, \delta \underline{U}^k = \underline{Q}^{k+\frac{1}{2}}\, \Delta_+ \underline{U}^k + \Delta t\, \underline{F}^{k+\frac{1}{2}}, \\ B(U_b^{k+1}, Z^{k+1}, t^{k+1}) = 0, \\ \qquad k=0,\frac{1}{2},\cdots,T/\Delta t - 1. \end{cases} \tag{2.14}$$

Here $\delta_{\frac{1}{2}} \underline{U}^0 = \underline{U}^{\frac{1}{2}} - \underline{U}^0$, $\underline{Q}$ may be defined in the same way as $\underline{S}$ is done, and $\underline{Q}$, $\underline{R}$, $\underline{S}$ satisfy the relation

$$\underline{SU} = \underline{RU} + \underline{Q}\, \Delta_+ \underline{U}. \tag{2.15}$$

It is clear that we can also construct a second order scheme in the following form

$$\begin{cases} \underline{R}_1^{k+\frac{1}{2}} \underline{U}^{k+1} = \underline{S}_1^{k+\frac{1}{2}} \underline{U}^k + \Delta t\, \underline{F}_1^{k+\frac{1}{2}}, \\ B(U_b^{k+1}, Z^{k+1}, t^{k+1}) = 0, \end{cases} \tag{2.16}$$

where $\underline{R}_1^{k+\frac{1}{2}}$, $\underline{S}_1^{k+\frac{1}{2}}$ and $\underline{F}_1^{k+\frac{1}{2}}$ depend on $\underline{U}^{k+1}$, Z^{k+1}, t^{k+1}, $\underline{U}^k$, Z^k, t^k.

3. BASIC ASSUMPTIONS

Consider the scalar equation

$$\frac{\partial u}{\partial t} + \lambda(u,\xi,t)\frac{\partial u}{\partial \xi} = 0, \quad 0 \le \xi \le 1, \quad 0 \le t \le T, \tag{3.1}$$

and suppose that it is approximated by difference equations of the following form

$$\sum_{h=h_1(m)}^{h_2(m)} \gamma_{h,m}^{k+\frac12} u_{m+h}^{k+1} = \sum_{h=h_1(m)}^{h_2(m)} s_{h,m}^{k+\frac12} u_{m+h}^{k}, \quad m \in \underline{g}(\lambda) \tag{3.2}$$

$$0 \le m+h_1(m) \le m + h_2(m) \le M, \quad \max_m\{|h_1(m)|, \ |h_2(m)|\} \le H, \quad k=0,1,\cdots .$$

For scheme (3.2), besides the consistency condition, some stability condition and "well-conditioned" condition should be required. We suppose that the following "most weak" stability condition and "well-conditioned" condition are satisfied:

(1) The von Neumann condition

$$(\gamma_m^{k+\frac12}(\theta))^* \gamma_m^{k+\frac12}(\theta) - (s_m^{k+\frac12}(\theta))^* s_m^{k+\frac12}(\theta) \ge 0 \tag{3.3}$$

holds for any $m \in \underline{g}(\lambda)$, where

$$\gamma_m^{k+\frac12}(\theta) = \sum_{h=h_1(m)}^{h_2(m)} \gamma_{h,m}^{k+\frac12}(0,0)e^{ih\theta},$$

$$s_m^{k+\frac12}(\theta) = \sum_{h=h_1(m)}^{h_2(m)} s_{h,m}^{k+\frac12}(0,0)e^{ih\theta}$$

and $(\gamma_m^{k+\frac12}(\theta))^*$, $(s_m^{k+\frac12}(\theta))^*$ are conjugate complex numbers of $\gamma_m^{k+\frac12}(\theta)$, $s_m^{k+\frac12}(\theta)$, $\gamma_{h,m}^{k+\frac12}(0,0)$, $s_{h,m}^{k+\frac12}(0,0)$ being the linear main parts of $\gamma_{h,m}^{k+\frac12} \equiv \gamma_{h,m}^{k+\frac12}(\Delta\xi, \Delta t) = \gamma_h(m\Delta\xi, (k+\frac12)\Delta t, \Delta\xi, \Delta t)$, $s_{h,m}^{k+\frac12} \equiv s_{h,m}^{k+\frac12}(\Delta\xi, \Delta t) = s_h(m\Delta\xi, (k+\frac12)\Delta t, \Delta\xi, \Delta t)$.

(2) The 'well-conditioned' condition

$$(\gamma_m^{k+\frac12}(\theta))^* \gamma_m^{k+\frac12}(\theta) \ge c_1 > 0 \tag{3.4}$$

holds for any $m \in \underline{g}(\lambda)$, c_1 being a positive constant. For explicit schemes, (3.4) always holds.

(3) If u is Lipschitz continuous with respect to ξ and t, then $\gamma_{h,m}^{k+\frac12}$, $s_{h,m}^{k+\frac12}$ are Lipschitz continuous with respect to ξ and t, i.e.,

$$|\gamma_{h,m}^{k+\frac12} - \gamma_{h,m-1}^{k+\frac12}| \le c\Delta\xi, \qquad |s_{h,m}^{k+\frac12} - s_{h,m-1}^{k+\frac12}| \le c\Delta\xi \tag{3.5}$$

$$|\gamma_{h,m}^{k+\frac12} - \gamma_{h,m}^{k-\frac12}| \le c\Delta t, \qquad |s_{h,m}^{k+\frac12} - s_{h,m}^{k-\frac12}| \le c\Delta t$$

and the differences between $\gamma_{h,m}^{k+\frac12}(0,0)$; $s_{h,m}^{k+\frac12}(0,0)$ and $\gamma_{h,m}^{k+\frac12}$, $s_{h,m}^{k+\frac12}$ satisfy

$$|\gamma_{h,m}^{k+\frac12}(0,0) - \gamma_{h,m}^{k+\frac12}| \le c(\Delta t + \Delta\xi),$$
$$|s_{h,m}^{k+\frac12}(0,0) - s_{h,m}^{k+\frac12}| \le c(\Delta t + \Delta\xi), \tag{3.6}$$

where c is a constant. From now on, c_i denotes a constant, and we use c to express different constants if it is not necessary to give a specified subscript i.

We define

$$T^k = \sum_{m\in\underline{g}(\lambda)} \Big(\sum_h \gamma_{h,m}^{k+\frac12} u_{m+h}^k\Big)^2 \Delta\xi, \tag{3.7}$$

$$||u^k||^2 = \sum_{m=0}^{M} |u_m^k|^2 \Delta\xi, \tag{3.8}$$

i.e., $||u^k||$ is the L_2 norm of u^k.(In this paper, $||f||$ always means the L_2-norm of f .)

We say that scheme (3.2) possesses Property A if from (3.3)-(3.6) we can derive the two inequalities:

(i) $$T^k + c\Delta\xi||u^k||^2 \ge -c|u_0^k|^2 \Delta\xi\,\delta_0(\lambda_0^k) - c|u_M^k|^2\Delta\xi\,\delta_1(\lambda_M^k) + c_2\sum_{m\in\underline{g}(\lambda^k)} |u_m^k|^2\Delta\xi; \tag{3.9}$$

(ii) $$T^{k+1} - T^k \le [c\delta_0(\lambda_0^k) - c_2|\sigma_0^k|(1-\delta_0(\lambda_0^k))]|u_0^k|^2\Delta\xi + [c\delta_1(\lambda_M^k) - c_2|\sigma_M^k|(1-\delta_1(\lambda_M^k))]|u_M^k|^2\Delta\xi + c\Delta\xi||u^k||^2, \tag{3.10}$$

where

$$\delta_0(\lambda_0^k) \equiv \begin{cases} 1, & \lambda_0^k > 0 \\ 0 & \lambda_0^k \le 0 \end{cases}, \qquad \delta_1(\lambda_M^k) \equiv \begin{cases} 1, & \lambda_M^k < 0 \\ 0, & \lambda_M^k \ge 0 \end{cases},$$

c is a constant and c_2 is a positive constant. Actually, using $\delta_0(\lambda_0^k)$ and $\delta_1(\lambda_M^k)$, $\underline{g}(\lambda^k)$ can be expressed as follows

$$\underline{g}(\lambda^k) = \{\delta_0(\lambda_0^k), \ \delta_0(\lambda_0^k)+1, \cdots, M-\delta_1(\lambda_M^k)\}.$$

Therefore if $\underline{g}(\lambda^k)$ is the same for all k, then $\delta_0(\lambda_0^k)$, $\delta_1(\lambda_M^k)$ are also the same for all k. We have used $\underline{g}(\lambda)$ instead of $\underline{g}(\lambda^k)$. Consequently, in what follows, we use $\delta_0(\lambda_0)$, $\delta_1(\lambda_M)$ instead of $\delta_0(\lambda_0^k)$, $\delta_1(\lambda_M^k)$.

Suppose λ = constant > 0. Using scheme (2.5), we can obtain a system of difference equations of form (3.2) approximating (3.1) as follows:

$$\frac{u_m^{k+1} + u_{m-1}^{k+1}}{2} + \sigma\frac{u_m^{k+1} - u_{m-1}^{k+1}}{2} = \frac{u_m^{k} + u_{m-1}^{k}}{2} - \sigma\frac{u_m^{k} - u_{m-1}^{k}}{2}, \qquad m=1,2,\cdots,M. \tag{3.11}$$

Therefore

$$T^k = [\sum_{m=1}^{M} \frac{1}{4}((1+\sigma)u_m + (1-\sigma)u_{m-1})^2 \;]^k \Delta\xi$$

$$= [\frac{1}{4}(1-\sigma)^2 u_0^2 + \frac{1+\sigma^2}{2}\sum_{m=1}^{M-1} u_m^2 + \frac{(1+\sigma)^2}{4} u_M^2$$

$$+ \frac{1}{2}(1-\sigma^2)\sum_{m=1}^{M} u_{m-1} u_m]^k \Delta\xi$$

$$= \begin{cases} [\frac{1}{4}((1-\sigma)^2-(1-\sigma^2))u_0^2 + \sigma^2\sum_{m=1}^{M-1} u_m^2 + \frac{1}{4}((1+\sigma)^2 - (1-\sigma^2))u_M^2 \\ \quad + \frac{1}{2}(1-\sigma^2)(\frac{1}{2}u_0^2 + \sum_{m=1}^{M-1} u_m^2 + \frac{1}{2}u_M^2 + \sum_{m=1}^{M} u_{m-1}u_m)]^k\Delta\xi \\ \quad \geq [\frac{1}{2}\sigma(\sigma-1)u_0^2 + \sigma^2\sum_{m=1}^{M} u_m^2]^k\Delta\xi, \text{ if } 0 \leq \sigma \leq 1, \\ [\frac{1}{4}((1-\sigma)^2-(\sigma^2-1))u_0^2 + \sum_{m=1}^{M-1} u_m^2 + \frac{1}{4}((1+\sigma)^2 - (\sigma^2-1))u_M^2 \\ \quad + \frac{1}{2}(\sigma^2-1)(\frac{1}{2}u_0^2 + \sum_{m=1}^{M-1} u_m^2 + \frac{1}{2}u_M^2 - \sum_{m=1}^{M} u_{m-1}u_m)]^k\Delta\xi \\ \quad \geq [\frac{1}{2}(1-\sigma)u_0^2 + \sum_{m=1}^{M} u_m^2]^k\Delta\xi, \text{ if } 1 \leq \sigma \end{cases} \tag{3.12}$$

and

$$T^{k+1} - T^k = [\sum_{m=1}^{M} \frac{1}{4}((1-\sigma)u_m + (1+\sigma)u_{m-1})^2$$

$$- \sum_{m=1}^{M} \frac{1}{4}((1+\sigma)u_m - (1-\sigma)u_{m-1})^2]^k \Delta\xi$$

$$= [\sum_{m=1}^{M} \frac{1}{4}((1-\sigma)^2 - (1+\sigma)^2)u_m^2 + \frac{1}{4}\sum_{m=0}^{M-1}((1+\sigma)^2-(1-\sigma)^2)u_m^2]^k \Delta\xi$$

$$= [\sigma u_0^2 - \sigma u_M^2]^k \Delta\xi, \tag{3.13}$$

where $[\ldots]^k$ means that every quantity in $[\cdots]$ has a superscript k. Let $c = \max\{\frac{1}{2}\sigma(1-\sigma), \frac{1}{2}(\sigma-1), \sigma\}$, $c_2 = \min\{\sigma^2, 1\}$, noticing $\delta_0(\lambda_0) = 1$ and $\delta_1(\lambda_M) = 0$ in the present case, we can write (3.12) and (3.13) in the form of (3.9) and (3.10) respectively if $0 < \sigma$. For scheme (2.5), if $\lambda > 0$, (3.3) always holds and (3.4) is equivalent to $0 < \sigma$. Therefore, scheme (2.5) has Property A if λ = constant > 0.

The following scheme approximating (3.1) with λ = constant ≥ 0,

$$u_m^{k+1} = u_m^k - \sigma(u_m^k - u_{m-1}^k), \qquad m = 1,2,\cdots,M \tag{3.14}$$

is also in the form (3.2). In this case

$$T^k = \sum_{m=1}^{M} |u_m^k|^2 \Delta\xi$$

and if $0 \leq \sigma \leq 1$, we have

$$T^{k+1} - T^k = [\sum_{m=1}^{M} ((1-\sigma)u_m + \sigma u_{m-1})^2 - \sum_{m=1}^{M} u_m^2]^k \Delta\xi$$

$$=[(1-\sigma)^2\sum_{m=1}^{M} u_m^2 + \sigma^2\sum_{m=0}^{M-1} u_m^2 + 2(1-\sigma)\sigma\sum_{m=1}^{M} u_m u_{m-1} - \sum_{m=1}^{M} u_m^2]^k\Delta\xi$$

$$\leq[(1-\sigma)^2\sum_{m=1}^{M} u_m^2 + \sigma^2\sum_{m=0}^{M-1} u_m^2 + (1-\sigma)\sigma(u_0^2 + 2\sum_{m=1}^{M-1} u_m^2 + u_M^2) - \sum_{m=1}^{M} u_m^2]^k\Delta\xi$$

$$= [\sigma u_0^2 - \sigma u_M^2]^k \Delta\xi.$$

Let $c = \sigma$, $c_2=1$, they can be written in forms (3.9) and (3.10). For (3.14) the condition (3.3) is equivalent to $0 \leq \sigma \leq 1$ and (3.4) always holds. That is, (3.14) also possesses Property A if λ= constant ≥ 0. Actually, many schemes are of Property A even for the case $\lambda = \lambda(\xi,t)$. In fact, for several schemes with variable coefficients, including schemes (2.5), (2.6), (2.7) and some combinations of them, we have proved[3] that they possess Property A. Some of results are given in Section 6.

As we have done in Section 2, the boundary conditions can be written in the following form

$$B(U_b, Z, t) = 0. \tag{3.15}$$

Its first variation equation is

$$\frac{\partial B}{\partial(U_b,Z)}\begin{bmatrix}\delta U_b \\ \delta Z\end{bmatrix} = 0.$$

We further rewrite it as

$$B_g \begin{bmatrix}G_b & \delta U_b \\ & \delta Z\end{bmatrix} = 0, \tag{3.16}$$

where

$$G_b = \begin{bmatrix} G_{1,0} & & & \\ & G_{1,M} & & \\ & & \ddots & \\ & & & G_{L,0} \\ & & & & G_{L,M} \end{bmatrix} \tag{3.17}$$

and

$$B_g = \frac{\partial B}{\partial(U_b, Z)} \begin{bmatrix} G_b^{-1} & \\ & I_{L+1} \end{bmatrix}, \tag{3.18}$$

I_{L+1} being an $(L + 1) \times (L + 1)$ unit matrix. Clearly $G_b \delta U_b$ is a 2LN-vector whose components are $(G_n^* \delta U)_{\ell,0}$ and $(G_n^* \delta U)_{\ell,M}$, $n=1,2,\cdots,N$, $\ell = 1,2,\cdots,L$. $(G_b \delta U_b)_i$ denotes the i-th component of $G_b \delta U_b$. Λ_b is defined as follows: its i-th component λ_{bi} is $\lambda_{n,\ell,0}$ if $(G_b \delta U_b)_i = (G_n^* \delta U)_{\ell,0}$ or $\lambda_{n,\ell,M}$ if $(G_b \delta U_b)_i = (G_n^* \delta U)_{\ell,M}$. The set $I = \{1,2,\cdots,2LN\}$ is divided into I_0 and I_1 in the following way: in the case $\lambda_{bi} = \lambda_{n,\ell,0}$ for some i, we say that i belongs to I_0 if $\delta_0(\lambda_{bi}) = \delta_0(\lambda_{n,\ell,0}) = 0$ and to I_1 if $\delta_0(\lambda_{bi}) = \delta_0(\lambda_{n,\ell,0}) = 1$; in the case $\lambda_{bi} = \lambda_{n,\ell,M}$ for some i, we say that i belongs to I_0 if $\delta_1(\lambda_{bi}) = \delta_1(\lambda_{n,\ell,M}) = 0$, and to I_1 if $\delta_1(\lambda_{bi}) = \delta_1(\lambda_{n,\ell,M}) = 1$.

We say the boundary condition (3.15) possesses Property B if for any V_b, Y satisfying

$$B_g \begin{pmatrix} V_b \\ Y \end{pmatrix} = E, \tag{3.19}$$

the following inequality holds:

$$|Y|^2 + \sum_{i \in I_1} |v_{bi}|^2 \le c\Big(\sum_{i \in I_0} |\lambda_{bi}| |v_{bi}|^2 + |E|^2 \Big), \tag{3.20}$$

where c is a constant and v_{bi} is the i-th component of V_b.

In practical problems we usually have such an inequality. From the definitions of $\delta_0(\lambda_{n,\ell,0})$ and $\delta_1(\lambda_{n,\ell,M})$ we know that if $i \in I_0$, the characteristics line corresponding to $\lambda_{bi} = \lambda_{n,\ell,0}$ or $\lambda_{n,\ell,M}$ arrives at a boundary. This means that if $i \in I_0$, the value of v_{bi} should be obtained from partial differential equations. Therefore, when all such v_{bi} are given, the boundary conditions should be able to give all values at boundaries. That is, the system

$$\begin{cases} B_g \begin{pmatrix} V_b \\ Y \end{pmatrix} = E \\ V_b^{(0)} = E^{(0)} \end{cases} \tag{3.21}$$

should have a solution, where $V_b^{(0)}$ denotes the vector consisting of all the v_{bi}, $i \in I_0$ and $E^{(0)}$ is a given vector. If (3.21) has a unique solution then we have

$$|Y|^2 + \sum_{i \in I_1} |v_{bi}|^2 \le c\Big(\sum_{i \in I_0} |v_{bi}|^2 + |E|^2 \Big). \tag{3.22}$$

Moreover, if all $|\lambda_{bi}| \ge$ a positive constant, then (3.22) can be changed to (3.20)

$\lambda_{bi^*} = 0$ for certain i^* and the i^*-th column of B_g is a zero-vector, which happens in several practical problems, then we will have

$$|Y|^2 + \sum_{i \in I_1} |v_{bi}|^2 \le c\Big[\sum_{i \in I_0 \text{ but } i \ne i^*} |v_{bi}|^2 + |E|^2 \Big]. \tag{3.23}$$

(3.23) can also be changed to (3.20). Therefore, (3.20) holds for many practical problems.

For the coefficients and the nonhomogeneous terms of differential equations and difference equations and boundary conditions, we assume that they are Lipschitz continuous with respect to their arguments. For the exact solution, besides the Lipschitz continuity in every subregion, we further assume that the truncation error will really be $0(\Delta t^3)$ for second order schemes.

4. EXISTENCE OF SOLUTIONS OF DIFFERENCE EQUATIONS

In this section, we shall prove that if

(i) (3.9) and (3.22) hold;

(ii) the errors at $t = k\Delta t$, $(k + 1/2)\Delta t$ are $0(\Delta t^2)$;

(iii) the coefficients in difference equations and the functions in boundary conditions are Lipschitz continuous with respect to their arguments,

then (2.11) has a solution $\{\underline{U}^{k+1}, Z^{k+1}\}$, and the difference between the solution and the exact solution is $0(\Delta t^2)$.

Suppose $\tilde{\underline{U}}, \tilde{Z}$ be the exact solution and $\tilde{U}_b$ is the equivalent of U_b in the case of exact solution. Since (2.11) is a second order scheme, we have

$$\begin{cases} \tilde{\underline{R}}^{k+\frac12} \tilde{\underline{U}}^{k+1} = \tilde{\underline{S}}^{k+\frac12} \tilde{\underline{U}}^{k} + \Delta t\, \tilde{\underline{F}}^{k+\frac12} + 0(\Delta t^3) \\ B(\tilde{U}_b^{k+1}, \tilde{Z}^{k+1}, t^{k+1}) = 0, \quad k = 0, 1/2, \cdots, T/\Delta t - 1, \end{cases} \tag{4.1}$$

where $\tilde{\underline{R}}$, $\tilde{\underline{S}}$, $\tilde{\underline{F}}$ are almost the same as $\underline{R}$, $\underline{S}$, $\underline{F}$, but arguments $\underline{U}$, Z are substituted by $\tilde{\underline{U}}$, $\tilde{Z}$, and $0(\Delta t^3)$ denotes a vector whose L_2 norm is of order Δt^3. Noticing (2.15), we can rewrite the first equation of (4.1) as

$$\begin{aligned} \underline{R}^{k+\frac12} \tilde{\underline{U}}^{k+1} &= (\underline{R}^{k+\frac12} - \tilde{\underline{R}}^{k+\frac12}) \tilde{\underline{U}}^{k+1} + \tilde{\underline{S}}^{k+\frac12} \tilde{\underline{U}}^{k} + \Delta t\, \tilde{\underline{F}}^{k+\frac12} + 0(\Delta t^3) \\ &= \underline{S}^{k+\frac12} \underline{U}^{k} + \Delta t\, \underline{F}^{k+\frac12} + (\underline{R}^{k+\frac12} - \tilde{\underline{R}}^{k+\frac12}) \tilde{\underline{U}}^{k+1} \\ &\quad - \underline{S}^{k+\frac12} (\underline{U}^{k} - \tilde{\underline{U}}^{k}) - (\underline{S}^{k+\frac12} - \tilde{\underline{S}}^{k+\frac12}) \tilde{\underline{U}}^{k} - \Delta t (\underline{F}^{k+\frac12} - \tilde{\underline{F}}^{k+\frac12}) + 0(\Delta t^3) \\ &= \underline{S}^{k+\frac12} \underline{U}^{k} + \Delta t\, \underline{F}^{k+\frac12} + (\underline{R}^{k+\frac12} - \tilde{\underline{R}}^{k+\frac12}) \delta \tilde{\underline{U}}^{k} \\ &\quad - (\underline{Q}^{k+\frac12} - \tilde{\underline{Q}}^{k+\frac12}) \Delta_+ \tilde{\underline{U}}^{k} - \underline{S}^{k+\frac12} (\underline{U}^{k} - \tilde{\underline{U}}^{k}) - \Delta t (\underline{F}^{k+\frac12} - \tilde{\underline{F}}^{k+\frac12}) + 0(\Delta t^3). \end{aligned}$$

According to the given condition, the L_2 norms of the errors $\begin{bmatrix} \underline{U}^k - \tilde{\underline{U}}^k \\ Z^k - \tilde{Z}^k \end{bmatrix}$ and $\begin{bmatrix} \underline{U}^{k+\frac12} - \tilde{\underline{U}}^{k+\frac12} \\ Z^{k+\frac12} - \tilde{Z}^{k+\frac12} \end{bmatrix}$ are $0(\Delta t^2)$, so their L_∞ norms are $0(\Delta t^{3/2})$. Moreover, in each row has only several arguments. Therefore ,

$$\Big|\Big| (\underline{R}^{k+\frac{1}{2}}-\underline{\tilde{R}}^{k+\frac{1}{2}})\delta\underline{\tilde{U}}^{k} - (\underline{Q}^{k+\frac{1}{2}} - \underline{\tilde{Q}}^{k+\frac{1}{2}})\Delta_{+}\underline{\tilde{U}}^{k}-\underline{S}^{k+\frac{1}{2}}(\underline{U}^{k}-\underline{\tilde{U}}^{k}) - \Delta t(\underline{F}^{k+\frac{1}{2}} - \underline{\tilde{F}}^{k+\frac{1}{2}}) + 0(\Delta t^3)\Big|\Big| \quad \text{is } 0(\Delta t^2).$$

Consequently, (4.1) can be written in the form

$$\begin{cases} \underline{R}^{k+\frac{1}{2}}\,\underline{\tilde{U}}^{k+1} = \underline{S}^{k+\frac{1}{2}}\,\underline{U}^{k} + \Delta t\,\underline{F}^{k+\frac{1}{2}} + 0(\Delta t^2), \\ B(\tilde{U}_b^{k+1}, \tilde{Z}^{k+1}, t^{k+1}) = 0. \end{cases} \tag{4.2}$$

Consider the following system

$$\underline{A}(\underline{X}) = \underline{F} \tag{4.3}$$

where

$$\underline{X} = \begin{bmatrix} \underline{U} \\ Z \end{bmatrix}, \quad \underline{A}(\underline{X}) = \begin{bmatrix} \underline{R}^{k+\frac{1}{2}}\,\underline{U} \\ B(U_b, Z, t^{k+1}) \end{bmatrix}.$$

Here the numbers of components of B, U_b, Z are fixed, but the number of components of $\underline{U}$ is $0(\frac{1}{\Delta t})$. Also, we suppose every element of $\frac{\partial B}{\partial (U_b, Z)}$ is Lipschitz contin - uous with respect to its arguments. From (4.2) we know that if

$$\underline{F} = \underline{F}^* \equiv \begin{bmatrix} \underline{S}^{k+\frac{1}{2}}\,\underline{U}^{k} + \Delta t\,\underline{F}^{k+\frac{1}{2}} + 0(\Delta t^2) \\ 0 \end{bmatrix},$$

(4.3) has a solution $\underline{X}^* = \begin{bmatrix} \underline{\tilde{U}}^{k+1} \\ \tilde{Z}^{k+1} \end{bmatrix}$. Now we shall prove that if $||\underline{F}-\underline{F}^*|| \le c\Delta t^{\frac{1}{2}+\delta}$, $\delta > 0$ and Δt is small enough, then (4.3) has a solution which satisfies

$$||\underline{X} - \underline{X}^*|| \le 2||\underline{A}'^{-1}(\underline{X}^*)||\ ||\underline{F} - \underline{F}^*|| \tag{4.4}$$

and the solution of (4.3) satisfying (4.4) is unique, where $\underline{A}'$ is the Jacobian of $\underline{A}$ with respect to $\underline{X}$.

First, we would like to point out that because the numbers of components of B, U_b, Z are fixed and every element of $\frac{\partial B}{\partial(U_b, Z)}$ is Lipschitz continuous with respect to its arguments, we have

$$||\underline{A}'(\underline{X}_1) - \underline{A}'(\underline{X}_2)|| \le c||\underline{X}_1 - \underline{X}_2||_{L_\infty}. \tag{4.5}$$

Consider the simplified Newton iteration with $\underline{X}^0 = \underline{X}^*$:

$$\underline{X}^{k+1} = \underline{X}^{k} + \underline{A}'^{-1}(\underline{X}^*)(\underline{F} - \underline{A}(\underline{X}^{k})), \quad k = 0,1,\cdots. \tag{4.6}$$

From (4.6) we have

$$\begin{aligned} \underline{X}^{k+1}-\underline{X}^* &= \underline{X}^{k}-\underline{X}^{*}+\underline{A}'^{-1}(\underline{X}^*)(\underline{A}(\underline{X}^*) - \underline{A}(\underline{X}^{k}))+\underline{A}'^{-1}(\underline{X}^*)(\underline{F} - \underline{F}^{*}) \\ &= \underline{A}'^{-1}(\underline{X}^*)(\underline{A}'(\underline{X}^*) - \underline{A}'(\underline{X}^* + \xi(\underline{X}^{k} - \underline{X}^{*})))(\underline{X}^{k}- \underline{X}^*) \end{aligned}$$

Therefore, if (4.4) holds for $\underline{X}^k$ and Δt is small enough, then (4.4) holds also for $\underline{X}^{k+1}$. In fact, when (4.4) holds for $\underline{X}^k$, we have

$$||\underline{X}^k - \underline{X}^*||_{L_\infty} \le c\ \Delta t^{\delta}.$$

Therefore, letting Δt be small enough, from (4.5) we obtain

$$||\underline{A}'(\underline{X}^*) - \underline{A}'(\underline{X}^* + \xi(\underline{X}^k - \underline{X}^*))|| \le \frac{1}{2||\underline{A}'^{-1}(\underline{X}^*)||} \tag{4.8}$$

and

$$||\underline{X}^{k+1} - \underline{X}^{*}|| \le 2||\underline{A}'^{-1}(\underline{X}^*)||\ ||\underline{F} - \underline{F}^*||$$

can be derived from (4.7). Noting

$$||\underline{X}^1 - \underline{X}^*|| \le ||\underline{A}'^{-1}(\underline{X}^*)||\ ||\underline{F} - \underline{F}^*||,$$

and using the inductive method, we know (4.4) holds for all X^k.

From (4.6) we have

$$\begin{aligned} \underline{X}^{k+2} - \underline{X}^{k+1} &= \underline{X}^{k+1}-\underline{X}^{k} + \underline{A}'^{-1}(\underline{X}^{*})(\underline{A}(\underline{X}^{k}) - \underline{A}(\underline{X}^{k+1})) \\ &= \underline{A}'^{-1}(\underline{X}^{*})(\underline{A}'(\underline{X}^{*}) - \underline{A}'(\underline{X}^{k} + \xi(\underline{X}^{k+1}-\underline{X}^{k})))(\underline{X}^{k+1}-\underline{X}^{k}), \\ & \qquad 0 \le \xi \le 1. \end{aligned}$$

Since both $\underline{X}^{k+1}$ and $\underline{X}^{k}$ satisfy (4.4), we can choose a small Δt such that

$$||\underline{A}'(\underline{X}^*)-\underline{A}'(\underline{X}^{k}+\xi(\underline{X}^{k+1}-\underline{X}^{k}))|| \le \frac{1}{2||\underline{A}'^{-1}(\underline{X}^*)||}. \tag{4.9}$$

Therefore we have

$$||\underline{X}^{k+2} - \underline{X}^{k+1}|| \le \frac{1}{2}||\underline{X}^{k+1} - \underline{X}^{k}||,$$

and the convergence of the iteration (4.6) can be obtained immediately.

Suppose both $\underline{X}_1$ and $\underline{X}_2$ are solutions of (4.3) and satisfy (4.4). Noting that in this case

$$||\underline{A}'(\underline{X}^*)-\underline{A}'(\underline{X}_1 + \xi(\underline{X}_2 - \underline{X}_1))|| \le \frac{1}{2||\underline{A}'^{-1}(\underline{X}^*)||},$$

we can have

$$\begin{aligned} ||\underline{A}'^{-1}(\underline{X}_1+ \xi(\underline{X}_2-\underline{X}_1))|| &\le ||(I-\underline{A}'^{-1}(\underline{X}^*)(\underline{A}'(\underline{X}^*)-\underline{A}'(\underline{X}_1+\xi(\underline{X}_1-\underline{X}_2)))^{-1}||\ ||\underline{A}'^{-1}(\underline{X}^{*})|| \\ &\le \frac{||A'^{-1}(X^*)||}{1-||\underline{A}'^{-1}(\underline{X}^*)||\ ||\underline{A}'(\underline{X}^*)-\underline{A}'(\underline{X}_1+\xi(\underline{X}_1-\underline{X}_2))||} \\ &\le 2||\underline{A}'^{-1}(\underline{X}^{*})||. \end{aligned}$$

Moreover, there is the inequality

$$||\underline{X}_1-\underline{X}_2|| \le ||\underline{A}'^{-1}(\underline{X}_1+\xi(\underline{X}_2 - \underline{X}_1))||\ ||\underline{A}(\underline{X}_1) - \underline{A}(\underline{X}_2)||.$$

Therefore, from $\underline{A}(\underline{X}_1) = \underline{A}(\underline{X}_2) = \underline{F}$ we can obtain $\underline{X}_1 = \underline{X}_2$, i.e., (4.3) have only one solution which satisfies (4.4).

Since both (4.2) and (2.11) are in the form (4.3) and the difference between the L_2 norm of the matrix

$$\underline{A}'^{-1} \equiv \left(\frac{\partial}{\partial(\underline{U},Z)} \begin{bmatrix} \underline{R}^{k+\frac{1}{2}}\underline{U} \\ B(U_b,Z,t^{k+1}) \end{bmatrix} \right)^{-1} \Bigg|_{\underline{U}=\tilde{\underline{U}}^{k+1},\ Z=\tilde{Z}^{k+1}} \tag{4.10}$$

can be shown, our conclusion is proved.

Let us consider the system

$$\begin{cases} \underline{R}^{k+\frac{1}{2}}\underline{U}^{k+1} = \underline{S}^{k+\frac{1}{2}}\underline{U}^k, \\ \dfrac{\partial \tilde{B}^{k+1}}{\partial(U_b,Z)} \begin{pmatrix} U_b^{k+1} \\ Z^{k+1} \end{pmatrix} = E, \end{cases} \tag{4.11}$$

where

$$\frac{\partial \tilde{B}^{k+1}}{\partial(U_b,Z)} = \frac{\partial B(U_b,Z,t)}{\partial(U_b,Z)} \Bigg|_{U_b=\tilde{U}_b^{k+1},\ Z=\tilde{Z}^{k+1},\ t=t^{k+1}}.$$

It is easy to know that if when a scheme is used to

$$\frac{\partial u}{\partial t} + \lambda \frac{\partial u}{\partial x} = 0, \quad 0 \le x \le 1, \tag{4.12}$$

we get a system in the form

$$\sum_h \gamma_{h,m}^{k+\frac{1}{2}} u_{m+h}^{k+1} = \sum_h s_{h,m}^{k+\frac{1}{2}} u_{m+h}^k, \quad m \in \underline{g}(\lambda),$$

then when the same scheme is used to

$$G_n^* \frac{\partial U}{\partial t} + \lambda_n G_n^* \frac{\partial U}{\partial \xi} = 0, \quad n = 1,2,\cdots,N,$$

for each $\lambda_{n,\ell}$ the system of difference equations

$$\sum_h R_{h,n,\ell,m}^{k+\frac{1}{2}} U_{\ell,m+h}^{k+1} = \sum_h S_{h,n,\ell,m}^{k+\frac{1}{2}} U_{\ell,m+h}^k, \quad m \in \underline{g}(\lambda_{n,\ell}) \tag{4.13}$$

can be rewritten in the form

$$\sum_h \left(\gamma_{h,n,\ell,m}^{k+\frac{1}{2}} u_{n,\ell,m+h}^{k+1} + 0(\Delta t)\, U_{\ell,m+h}^{k+1}\right) = \sum_h \left(s_{h,n,\ell,m}^{k+\frac{1}{2}} u_{n,\ell,m+h}^{k} + 0(\Delta t)\, U_{\ell,m+h}^{k}\right), \quad m \in \underline{g}(\lambda_{n,\ell}), \tag{4.14}$$

where u_n is the n-th component of $\tilde{G}U$, each $0(\Delta t)$ is an $N \times N$ matrix and its components are quantites of order Δt.

Define

$$\begin{cases} c_3^{m/M}, & \text{if } \delta_0(\lambda_{n,\ell}) = 1 \text{ and } \delta_1(\lambda_{n,\ell}) = 0, \\ c_3^{1-m/M}, & \ldots \end{cases}$$

$$b_{n,\ell,m} = \begin{cases} c_3, & \text{if } \delta_0(\lambda_{n,\ell}) = 0 \text{ and } \delta_1(\lambda_{n,\ell}) = 0, \\ 1, & \text{if } \delta_0(\lambda_{n,\ell}) = 1 \text{ and } \delta_1(\lambda_{n,\ell}) = 1, \end{cases} \tag{4.15}$$

where c_3 is a constant greater than 1. Clearly, on boundaries, if $i \in I_0$, the corresponding $b_{n,\ell,m} = c_3$ and if $i \in I_1$, $b_{n,\ell,m} = 1$.

Multiplying (4.14) by $b_{n,\ell,m}$, defining

$$\hat{u}_{n,\ell,m} = b_{n,\ell,m} u_{n,\ell,m} \text{ and noticing } b_{n,\ell,m}/b_{n,\ell,m+h} = 1 + 0(\Delta t),$$

we obtain

$$\sum_h \left(\gamma_{h,n,\ell,m}^{k+\frac{1}{2}} \hat{u}_{n,\ell,m+h}^{k+1} + 0(\Delta t)\, \hat{U}_{\ell,m+h}^{k+1}\right) = \sum_h \left(s_{h,n,\ell,m}^{k+\frac{1}{2}} \hat{u}_{n,\ell,m+h}^{k} + 0(\Delta t)\hat{U}_{\ell,m+h}^{k}\right), \quad m \in \underline{g}(\lambda_{n,\ell}), \tag{4.16}$$

where $\hat{U}$ denotes the vector $(\hat{u}_1, \hat{u}_2, \cdots, \hat{u}_N)^T$.

Through straightforward derivation, we have

$$T_{n,\ell}^{k+1} \equiv \sum_{m \in \underline{g}(\lambda_{n,\ell})} \left(\sum_h \gamma_{h,n,\ell,m}^{k+\frac{1}{2}} \hat{u}_{n,\ell,m+h}^{k+1} + 0(\Delta t)\hat{U}_{\ell,m+h}^{k+1}\right)^2 \Delta\xi$$

$$\ge \sum_{m \in \underline{g}(\lambda_{n,\ell})} \left(\sum_h \gamma_{h,n,\ell,m}^{k+\frac{1}{2}} \hat{u}_{n,\ell,m+h}^{k+1}\right)^2 \Delta\xi - c\,\Delta t\, \|\hat{\underline{U}}_\ell^{k+1}\|^2,$$

where $\|\hat{\underline{U}}_\ell\|^2 = \sum_{m=0}^{M} |\hat{U}_{\ell,m}|^2 \Delta\xi$ and c is a constant, $|\hat{U}_{\ell,m}|$ being the Euclidean norm of $\hat{U}_{\ell,m}$. Therefore, when (3.9) holds, we have

$$T_{n,\ell}^{k+1} \ge -c|\hat{u}_{n,\ell,0}^{k+1}|^2 \Delta\xi\, \delta_0(\lambda_{n,\ell,0}) - c|\hat{u}_{n,\ell,M}^{k+1}|^2 \Delta\xi\, \delta_1(\lambda_{n,\ell,M})$$

$$- c\,\Delta t\, \|\hat{\underline{U}}_\ell^{k+1}\|^2 + c_2 \sum_{m \in \underline{g}(\lambda_{n,\ell})} |\hat{u}_{n,\ell,m}^{k+1}|^2 \Delta\xi.$$

Consequently,

$$T_{a\ell\ell}^{k+1} \equiv \sum_{n=1}^{N} \sum_{\ell=1}^{L} T_{n,\ell}^{k+1} + c_3^2 \left| \frac{\partial \tilde{B}^{k+1}}{\partial(U_b,Z)} \begin{pmatrix} U_b^{k+1} \\ Z^{k+1} \end{pmatrix} \right|^2 \Delta\xi$$

$$\ge -c \sum_{i \in I_1} |\hat{u}_{bi}^{k+1}|^2 \Delta\xi + c_2 \sum_{n=1}^{N} \sum_{\ell=1}^{L} \sum_{m \in \underline{g}(\lambda_{n,\ell})} |\hat{u}_{n,\ell,m}^{k+1}|^2 \Delta\xi$$

$$+ c_3^2 |E|^2 \Delta\xi - c\,\Delta t\, \|\hat{\underline{U}}^{k+1}\|^2$$

$$\ge \frac{c_2}{2} \left(\|\hat{\underline{U}}^{k+1}\|^2 + |Z^{k+1}|^2 \Delta\xi \right)$$

$$-\left(c + \frac{c_2}{2}\right)\left(\sum_{i\in I_1} |u_{bi}^{k+1}|^2 + |Z^{k+1}|^2\right)\Delta\xi$$

$$+ \frac{c_2 c_3^2}{2} \sum_{i\in I_0} |u_{bi}^{k+1}|^2 \Delta\xi + c_3^2 |E|^2 \Delta\xi - c\Delta t \, ||\underline{\hat{U}}^{k+1}||^2 .$$

Here the following relations are used:

$$\left\{\begin{aligned}
&||\underline{\hat{U}}^{k+1}||^2 \equiv \sum_{n=1}^{N} \sum_{\ell=1}^{L} \sum_{m=0}^{M} |\hat{u}_{n,\ell,m}^{k+1}|^2 \Delta\xi \\
&\qquad = \sum_{n=1}^{N} \sum_{\ell=1}^{L} \sum_{m\in g(\lambda_{n,\ell})}' |\hat{u}_{n,\ell,m}^{k+1}|^2 \Delta\xi \\
&\qquad + \sum_{i\in I_1} |\hat{u}_{bi}^{k+1}|^2 \Delta\xi , \\
&\sum_{n=1}^{N} \sum_{\ell=1}^{L} \sum_{m\in \underline{g}(\lambda_{n,\ell})} |\hat{u}_{n,\ell,m}^{k+1}|^2 \Delta\xi \ge \sum_{i\in I_0} |\hat{u}_{bi}^{k+1}|^2 \Delta\xi, \\
&\sum_{i\in I_0} |\hat{u}_{bi}^{k+1}|^2 = c_3^2 \sum_{i\in I_0} |u_{bi}^{k+1}|^2 , \\
&\sum_{i\in I_1} |\hat{u}_{bi}^{k+1}|^2 = \sum_{i\in I_1} |u_{bi}^{k+1}|^2 .
\end{aligned}\right. \tag{4.17}$$

The second equation of (4.11) can be transformed into

$$\frac{\partial \tilde{B}^{k+1}}{\partial (U_b, Z)} \begin{pmatrix} \tilde{G}_b^{k+1} & \\ & I_{L+1} \end{pmatrix}^{-1} \begin{pmatrix} (\tilde{G}_b U_b)^{k+1} \\ Z^{k+1} \end{pmatrix} = E,$$

where $\tilde{G}_b$ is a matrix similar to G_b defined by (3.17), but U_b should be replaced by $\tilde{U}_b$. Therefore it has the form of (3.19) and the following inequality in the form (3.22) holds:

$$|Z|^2 + \sum_{i\in I_1} |u_{bi}|^2 \le c\left(\sum_{i\in I_0} |u_{bi}|^2 + |E|^2\right) . \tag{4.18}$$

Consequently, noticing $c_2 > 0$, we can choose such a c_3 that

$$T_{a\ell\ell}^{k+1} \ge \frac{c_2}{2}\left(||\underline{\hat{U}}^{k+1}||^2 + |Z^{k+1}|^2 \Delta\xi \right) - c\,\Delta t\, ||\underline{\hat{U}}^{k+1}||^2$$

$$\ge \frac{c_2}{4}\left(||\underline{\hat{U}}^{k+1}||^2 + |Z^{k+1}|^2 \Delta\xi \right). \tag{4.19}$$

Here suppose Δt be small enough. It is clear that

$$T_{a\ell\ell}^{k+1} = \left|\left| \underline{B} \begin{bmatrix} \underline{R}^{k+\frac{1}{2}} \; \underline{U}^{k+1} \\ \frac{\partial \tilde{B}^{k+1}}{\partial (U_b, Z)} \begin{bmatrix} U_b^{k+1} \\ Z^{k+1} \end{bmatrix} \end{bmatrix} \right|\right|^2 ,$$

where $\underline{B}$ is a diagonal matrix whose components are $b_{n,\ell,m}$, $n=1,2,\cdots,N$, $\ell=1,2,\cdots,L$, $m\in\underline{g}(\lambda_{n,\ell})$ and a certain number of c_3. From (4.19) we can have

$$||\underline{A}'^{-1}||^2 = \sup \frac{\left|\left|\begin{bmatrix} \underline{U} \\ Z \end{bmatrix}\right|\right|^2}{\left|\left|\underline{A}'\begin{bmatrix} \underline{U} \\ Z \end{bmatrix}\right|\right|^2} \le \sup \frac{||\underline{\hat{U}}^{k+1}||^2 + |Z^{k+1}|^2 \Delta\xi}{||\underline{B}||^{-2}\, T_{a\ell\ell}^{k+1}}$$

$$\le \frac{4}{c_2 ||\underline{B}||^{-2}} = \frac{4c_3^2}{c_2} . \tag{4.20}$$

This means we complete our proof.

It is not difficult to show the following results.

(1) If the conditions described at the beginning of this section hold, the errors of approximate solution at $t=(k+1+\delta)\Delta t$, $\delta=0, 1/2,\cdots,m/2$, m being a finite integer, will be $O(\Delta t^2)$.

(2) If the term $O(\Delta t^3)$ in (4.1) is changed to $O(\Delta t^2)$, the conclusion is still correct. Therefore there exists a solution $\begin{pmatrix} U^{\frac{1}{2}} \\ Z^{\frac{1}{2}} \end{pmatrix}$ of (2.10) satisfying the inequality

$$\left|\left| \begin{pmatrix} \underline{U}^{\frac{1}{2}} - \underline{\tilde{U}}^{\frac{1}{2}} \\ Z^{\frac{1}{2}} - \tilde{Z}^{\frac{1}{2}} \end{pmatrix} \right|\right| \le c\Delta t^2, \tag{4.21}$$

where c is a certain constant.

(3) For first order schemes, a similar conclusion is correct. The only difference is that the errors at the 'given' level(s) and at the 'unknown' level are $O(\Delta t)$.

(4) For scheme (2.16), if conditions (i) and (iii) are fulfilled and

(ii') the error at $t = k\Delta t$ is $O(\Delta t^2)$,

then the system (2.16) has a soluton $\{\underline{U}^{k+1}, Z^{k+1}\}$, and the difference between $\{\underline{U}^{k+1}, Z^{k+1}\}$ and the exact solution $\{\underline{\tilde{U}}^{k+1}, \tilde{Z}^{k+1}\}$ is $O(\Delta t^{3/2})$.

5. PROOF OF CONVERGENCE

As in Section 4, $\{\underline{\tilde{U}}, \tilde{Z}\}$ denotes the exact solution and suppose in each subregion $\{\underline{\tilde{U}}, \tilde{Z}\}$ is quite smooth so that $\{\underline{\tilde{U}}, \tilde{Z}\}$ satisfy (4.1). By means of (2.15), (4.1) can be rewritten as

$$\left\{\begin{aligned}
&\underline{\tilde{R}}^{k+\frac{1}{2}}\, \delta\, \underline{\tilde{U}}^{k} = \underline{\tilde{Q}}^{k+\frac{1}{2}}\, \Delta_+ \underline{\tilde{U}}^{k} + \Delta t\, \underline{\tilde{F}}^{k+\frac{1}{2}} + O_1^{k+\frac{1}{2}} (\Delta t^3), \\
&B(\tilde{U}_b^{k+1}, \tilde{Z}^{k+1}, t^{k+1}) = 0, \quad k = 0, 1/2, 1, \cdots, T/\Delta t-1,
\end{aligned}\right. \tag{5.1}$$

where $\underline{\tilde{Q}}$ is equivalent to $\underline{Q}$, i.e., substituting $\underline{\tilde{U}}$, $\tilde{Z}$ for $\underline{U}$, Z in $\underline{Q}$, we can obtain

$$\begin{cases} \underline{R}^{k+\frac12}\delta(\underline{U}^k - \tilde{\underline{U}}^k) + (\underline{R}^{k+\frac12} - \tilde{\underline{R}}^{k+\frac12})\,\delta\,\tilde{\underline{U}}^k \\ = \underline{Q}^{k+\frac12}\Delta_+(\underline{U}^k - \tilde{\underline{U}}^k) + (\underline{Q}^{k+\frac12} - \tilde{\underline{Q}}^{k+\frac12})\,\Delta_+\tilde{\underline{U}}^k \\ \quad + \Delta t(\underline{F}^{k+\frac12} - \tilde{\underline{F}}^{k+\frac12}) + O_1^{k+\frac12}(\Delta t^3), \\ \frac{\partial B^{k+1}}{\partial(U_b, Z)} \begin{bmatrix} U_b^{k+1} - \tilde{U}_b^{k+1} \\ Z^{k+1} - \tilde{Z}^{k+1}\end{bmatrix} = 0, \quad k = 0, \frac12, 1, \cdots, \end{cases} \tag{5.2}$$

where $U_b = U_b^{k+1} + \xi\,(U_b^{k+1} - \tilde{U}_b^{k+1})$ in $\frac{\partial B^{k+1}}{\partial(U_b, Z)}$, $0 \le \xi \le 1$. Since each $U_{\ell,m}^{k+\frac12}$ or $U_{\ell,m}^k$ appears only in several rows of $\underline{R}^{k+\frac12}$ and each row of $\underline{R}^{k+\frac12}$ has only several nonzero components, $(\underline{R}^{k+\frac12} - \tilde{\underline{R}}^{k+\frac12})\,\delta\,\tilde{\underline{U}}^k$ has the following form

$$O(\Delta t)(\underline{U}^{k+\frac12} - \tilde{\underline{U}}^{k+\frac12}) + O^*(\Delta t)(Z^{k+\frac12} - \tilde{Z}^{k+\frac12})$$
$$+ O(\Delta t)(\underline{U}^k - \tilde{\underline{U}}^k) + O^*(\Delta t)(Z^k - \tilde{Z}^k).$$

Here $O(\Delta t)$ denotes a matrix in each row or each column of which there are only several non-zero components of order Δt. Therefore, the L_2 norm of $O(\Delta t)$ is less than $c\Delta t$, c being a constant. $O^*(\Delta t)$ denotes a matrix whose number of columns is $L + 1$ and whose each component is a quantity of order Δt. Consequently, the L_2 norm of $O^*(\Delta t)$ is less than $c\Delta t^{\frac12}$, c being a constant. Obviously, $(\underline{Q}^{k+\frac12} - \tilde{\underline{Q}}^{k+\frac12})\Delta_+\tilde{\underline{U}}^k$ and $\Delta t(\underline{F}^{k+\frac12} - \tilde{\underline{F}}^{k+\frac12})$ have the same form. Therefore, the first part of (5.2) can be written in the form

$$\underline{R}^{k+\frac12}(\underline{U}^{k+1} - \tilde{\underline{U}}^{k+1}) = O(\Delta t)(\underline{U}^{k+\frac12} - \tilde{\underline{U}}^{k+\frac12}) + O^*(\Delta t)(Z^{k+\frac12} - \tilde{Z}^{k+\frac12})$$
$$+ R^{k+\frac12}(\underline{U}^k - \tilde{\underline{U}}^k) + \underline{Q}^{k+\frac12}\Delta_+(\underline{U}^k - \tilde{\underline{U}}^k)$$
$$+ O(\Delta t)(\underline{U}^k - \tilde{\underline{U}}^k) + O^*(\Delta t)(Z^k - \tilde{Z}^k) + O_1^{k+\frac12}(\Delta t^3).$$

Let $\underline{V}$ be a vector whose components are $\tilde{G}_{1,0}(U_{1,0} - \tilde{U}_{1,0}), \cdots, \tilde{G}_{1,M}(U_{1,M} - \tilde{U}_{1,M}), \cdots, \tilde{G}_{L,0}(U_{L,0} - \tilde{U}_{L,0}), \cdots, \tilde{G}_{L,M}(U_{L,M} - \tilde{U}_{L,M})$ from the top to the bottom, V_b be a vector whose components are $\tilde{G}_{1,0}(U_{1,0} - \tilde{U}_{1,0}), \tilde{G}_{1,M}(U_{1,M} - \tilde{U}_{1,M}), \cdots, \tilde{G}_{L,0}(U_{L,0} - \tilde{U}_{L,0})$ and $\tilde{G}_{L,M}(U_{L,M} - \tilde{U}_{L,M})$, and $Y = Z - \tilde{Z}$. Noticing (2.15), the error system can further be rewritten as

$$\begin{cases} \underline{R}_g^{k+\frac12}\ \underline{V}^{k+1} = O(\Delta t)\underline{V}^{k+\frac12} + O^*(\Delta t)\,Y^{k+\frac12} \\ \qquad + (\underline{S}_g^{k+\frac12} + O(\Delta t))\underline{V}^k + O^*(\Delta t)\,Y^k + O_1^{k+\frac12}(\Delta t^3), \\ B^{k+1}\begin{pmatrix} \underline{V}^{k+1} \\ \cdots \end{pmatrix} = 0, \end{cases} \tag{5.3}$$

where $\underline{R}_g^{k+\frac12} = \underline{R}^{k+\frac12}(\tilde{\underline{G}}^{k+1})^{-1}$, $\underline{S}_g^{k+\frac12} = \underline{S}^{k+\frac12}(\tilde{\underline{G}}^{k})^{-1}$,

$$\tilde{\underline{G}} \equiv \begin{bmatrix} \tilde{G}_{1,0} & & & & 0 \\ & \ddots & & & \\ & & \tilde{G}_{1,M} & & \\ & & & \ddots & \\ & & & & \tilde{G}_{L,0} \\ & & & & & \ddots \\ 0 & & & & & & \tilde{G}_{L,M} \end{bmatrix} \tag{5.4}$$

and $\underline{B}_g^{k+1}$ is defined by

$$\underline{B}_g^{k+1}\begin{pmatrix} \underline{V}^{k+1} \\ Y^{k+1} \end{pmatrix} \equiv B_g^{k+1}\begin{pmatrix} V_b^{k+1} \\ Y^{k+1} \end{pmatrix}, \tag{5.5}$$

B_g^{k+1} denoting $\frac{\partial B^{k+1}}{\partial(U_b, Z)}\begin{bmatrix} \tilde{G}_b^{k+1} & \\ & I_{L+1} \end{bmatrix}^{-1}$, $\tilde{G}_b \equiv \begin{bmatrix} \tilde{G}_{1,0} & & & & 0 \\ & \tilde{G}_{1,M} & & & \\ & & \ddots & & \\ & & & \tilde{G}_{L,0} & \\ 0 & & & & \tilde{G}_{L,M} \end{bmatrix}$. Rewriting the first equation of (5.3) as

$$\underline{R}_g^{k+\frac12}\ \underline{V}^{k+1} + O(\Delta t)\underline{V}^{k+\frac12} + O^*(\Delta t)Y^{k+\frac12}$$
$$= (\underline{S}_g^{k+\frac12} + O(\Delta t))V^k + O^*(\Delta t)Y^k + O_1^{k+\frac12}(\Delta t^3) \tag{5.6}$$

for $k = \frac12, \frac32, \cdots$, and combining (5.3) with $k = j$ and (5.6) with $k = j + \frac12$, j being an integer, then we have the final form of the error system

$$\hat{\underline{R}}^k\underline{W}^{k+1} = \hat{\underline{S}}^k\underline{W}^k + O^k(\Delta t^3), \tag{5.7}$$
$$k = 0, 1, \cdots, T/\Delta t - 1,$$

where

$$\begin{cases} \hat{\underline{R}}^k = \begin{bmatrix} \underline{R}_g^{k+1} & 0 & O(\Delta t) & O^*(\Delta t) \\ & \underline{B}_g^{k+3/2} & 0 & 0 \\ 0 & 0 & \underline{R}_g^{k+\frac12} & 0 \\ 0 & 0 & & \underline{B}_g^{k+1} \end{bmatrix} \\ \hat{\underline{S}}^k = \begin{bmatrix} \underline{S}_g^{k+1} + O(\Delta t) & O^*(\Delta t) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ O(\Delta t) & O^*(\Delta t) & \underline{S}_g^{k+\frac12} + O(\Delta t) & O^*(\Delta t) \\ 0 & 0 & 0 & 0 \end{bmatrix}, \end{cases} \tag{5.8}$$

$$\underline{W}^k = \begin{bmatrix} \underline{V}^{k+\frac{1}{2}} \\ Y^{k+\frac{1}{2}} \\ \underline{V}^k \\ Y^k \end{bmatrix}$$

$$0^k(\Delta t^3) = \begin{bmatrix} 0_1^{k+1}(\Delta t^3) \\ 0_1^{k+\frac{1}{2}}(\Delta t^3) \end{bmatrix} .$$

Therefore

$$\underline{W}^{k+1} = (\hat{\underline{R}}^k)^{-1}\, \hat{\underline{S}}^k \underline{W}^k + (\hat{\underline{R}}^k)^{-1}\, 0^k(\Delta t^3)$$

$$= (\hat{\underline{R}}^k)^{-1} \hat{\underline{S}}^k (\hat{\underline{R}}^{k-1})^{-1} \hat{\underline{S}}^{k-1} \underline{W}^{k-1} + (\hat{\underline{R}}^k)^{-1} \hat{\underline{S}}^k (\hat{\underline{R}}^{k-1})^{-1} 0^{k-1}(\Delta t^3)$$

$$+ (\hat{\underline{R}}^k)^{-1} 0^k(\Delta t^3)$$

$$= \prod_{j=0}^{k} (\hat{\underline{R}}^j)^{-1} \hat{\underline{S}}^j\, \underline{W}^0 + \sum_{i=0}^{k} \prod_{j=i+1}^{k} (\hat{\underline{R}}^j)^{-1} \hat{\underline{S}}^j (\hat{\underline{R}}^i)^{-1} 0^i(\Delta t^3).$$

Consequently, if

$$\| (\hat{\underline{R}}^i)^{-1} \| \leq c, \tag{5.9}$$

$$\| \prod_{j=j_1}^{j_2} (\hat{\underline{R}}^j)^{-1} \hat{\underline{S}}^j \| \leq c, \quad \text{for } 0 \leq j_1 \Delta t \leq j_2 \Delta t \leq T, \tag{5.10}$$

then

$$\| \underline{W}^{k+1} \| \leq c_4 (\Delta t^2 + \| \underline{W}^0 \|), \text{ for any } (k+1)\Delta t \leq T,$$

where c_4 is a certain constant. In Section 4 we pointed out $\left\| \begin{pmatrix} U^{\frac{1}{2}} - \tilde{U}^{\frac{1}{2}} \\ Z^{\frac{1}{2}} - \tilde{Z}^{\frac{1}{2}} \end{pmatrix} \right\| = 0(\Delta t^2)$, so $\| W^0 \| = 0(\Delta t^2)$. Therefore, noting the boundedness of $\| \tilde{\underline{G}}^{-1} \|$, we can obtain immediately

$$\left\| \begin{pmatrix} \underline{U} - \tilde{\underline{U}} \\ Z - \tilde{Z} \end{pmatrix}^{k+1} \right\| \leq c\Delta t^2, \; k=0, \tfrac{1}{2}, \cdots, T/\Delta t - 1, \tag{5.11}$$

where c is a constant.

That is, if the systems for $\underline{W}^{k+1}$

$$\hat{\underline{R}}^k \underline{W}^{k+1} = \hat{\underline{S}}^k \underline{W}^k, \qquad k = 0, 1, \cdots, \tag{5.12}$$

are "well-conditioned" and the procedure (5.12) is stable, which means that (5.9) and (5.10) hold respectively, the approximate solution $\{\underline{U}, Z\}$ obtained from (2.10)-(2.11) converges to the exact solution $\{\tilde{\underline{U}}, \tilde{Z}\}$.

In what follows, we shall discuss the stability of the procedure (5.12).

If we can find an invertible matrix $\underline{B}$ such that for the procedure (5.12)

$$\| \underline{B}\hat{\underline{R}}^k \underline{W}^{k+1} \|^2 - \| \underline{B}\hat{\underline{R}}^{k-1} \underline{W}^k \|^2 \leq c\Delta t \; \| W^k \|^2 \tag{5.13}$$

and

$$\max\{ \| \underline{B}\hat{\underline{R}}^k \|^2, \; \| (\underline{B}\hat{\underline{R}}^k)^{-1} \|^2 \} \leq c, \tag{5.14}$$

then we obtain

$$\| \underline{B}\hat{\underline{R}}^k \underline{W}^{k+1} \|^2 \leq \| \underline{B}\hat{\underline{R}}^{k-1} \underline{W}^k \|^2 + c\Delta t \| W^k \|^2$$

$$\leq (1 + c^2 \Delta t) \| \underline{B}\hat{\underline{R}}^{k-1} \underline{W}^k \|^2$$

$$\leq (1 + c^2 \Delta t)^{k+1-i} \| \underline{B}\hat{\underline{R}}^{i-1} \underline{W}^i \|^2$$

$$\leq e^{c^2 (k+1-i)\Delta t} \| \underline{B}\hat{\underline{R}}^{i-1} \underline{W}^i \|^2$$

$$\leq e^{c^2 (k+1-i)\Delta t} c \| \underline{W}^i \|^2 .$$

Therefrore

$$\| \underline{W}^{k+1} \|^2 \leq c^2 e^{c^2 (k+1-i)\Delta t} \| \underline{W}^i \|^2 .$$

Noticing

$$\underline{W}^{k+1} = \prod_{j=i}^{k} (\hat{\underline{R}}^j)^{-1} \hat{\underline{S}}^j \underline{W}^i,$$

we have

$$\| \prod_{j=i}^{k} (\hat{\underline{R}}^j)^{-1} \hat{\underline{S}}^j \|^2 \leq c^2 e^{c^2 (k+1-i)\Delta t} .$$

Moreoever, we can obtain (5.9) from (5.14) if $\underline{B}$ is invertible. Therefore, the proof of stability can be reduced to finding an invertible B such that (5.13) and (5.14) hold.

In Section 4, we point out that (4.13) can be rewritten in the form (4.16). Similarly, every equation in (5.12) which corresponds to some difference equation, after multiplying by a $b_{n,\ell,m}$ defined in (4.15), can be written in the following form

$$\sum_h (\gamma^{k+1}_{h,n,\ell,m} \bar{v}^{k+3/2}_{n,\ell,m+h} + 0(\Delta t)\bar{V}^{k+3/2}_{\ell,m+h} + 0(\Delta t)\bar{V}^{k+1}_{\ell,m+h}) + 0^*(\Delta t) Y^{k+1}$$

$$= \sum_h (s^{k+1}_{h,n,\ell,m} \bar{v}^{k+\frac{1}{2}}_{n,\ell,m+h} + 0(\Delta t)\, \bar{V}^{k+\frac{1}{2}}_{\ell,m+h}) + 0^*(\Delta t) Y^{k+\frac{1}{2}}, \tag{5.15}$$

which is in the form (5.6) or

$$\sum_h (\gamma^{k+\frac{1}{2}}_{h,n,\ell,m} \bar{v}^{k+1}_{n,\ell,m+h} + 0(\Delta t)\, \bar{V}^{k+1}_{\ell,m+h})$$

$$= \sum_h (s^{k+\frac{1}{2}}_{h,n,\ell,m} \bar{v}^{k}_{n,\ell,m+h} + 0(\Delta t)\bar{V}^{k}_{\ell,m+h} + 0(\Delta t)\, \bar{V}^{k+\frac{1}{2}}_{\ell,m+h})$$

$$+ 0^*(\Delta t)\, Y^k + 0^*(\Delta t)\, Y^{k+\frac{1}{2}}, \tag{5.16}$$

which is in the form (5.3).

Here $\bar{v}_{n,\ell,m} = b_{n,\ell,m} v_{n,\ell,m}$, $v_{n,\ell,m}$ being the n-th component of $V_{\ell,m} = \tilde{G}_{\ell,m}(U_{\ell,m} - \tilde{U}_{\ell,m})$, and $\bar{V}_{\ell,m} = (\bar{v}_{1,\ell,m}, \bar{v}_{2,\ell,m}, \cdots, \bar{v}_{N,\ell,m})^T$. Therefore, if B is a diagonal matrix whose diagonal element is $b_{n,\ell,m}$ when the corresponding equation is a difference equation related to λ_n at $\xi = \ell + m\Delta\xi$ or is c_3 when the corresponding equation is a boundary condition, then in

$$\underline{\hat{BR}}^k\ \underline{W}^{k+1} = \underline{\hat{BS}}^k\ \underline{W}^k$$

every difference equation has the form (5.15) or (5.16).

Through straightforward derivation, we have

$$\bar{T}^{k+3/2}_{n,\ell} \equiv \sum_{m\in\underline{g}(\lambda_{n,\ell})} (\sum_h (\gamma^{k+1}_{h,n,\ell,m}\ \bar{v}^{k+3/2}_{n,\ell,m+h} + 0(\Delta t)\bar{v}^{k+3/2}_{\ell,m+h} + 0(\Delta t)\bar{v}^{k+1}_{\ell,m+h})$$

$$+ 0^*(\Delta t)Y^{k+1})^2\Delta\xi$$

$$\geq \sum_{m\in\underline{g}(\lambda_{n,\ell})} (\sum_h \gamma^{k+1}_{h,n,\ell,m}\bar{v}^{k+3/2}_{n,\ell,m+h})^2\Delta\xi - c\Delta t\|\ \underline{\bar{V}}^{k+3/2}_{\ell}\|^2 - c\Delta t\|\ \underline{\bar{V}}^{k+1}_{\ell}\|^2$$

$$- c\Delta\xi\,|Y^{k+1}|^2 \tag{5.17}$$

and

$$T^{k+1}_{n,\ell} = \sum_{m\in\underline{g}(\lambda_{n,\ell})} (\sum_h (\gamma^{k+\frac{1}{2}}_{h,n,\ell,m}\ \bar{v}^{k+1}_{n,\ell,m+h} + 0(\Delta t)\bar{v}^{k+1}_{\ell,m+h}))^2\Delta\xi$$

$$\geq \sum_{m\in\underline{g}(\lambda_{n,\ell})} (\sum_h \gamma^{k+\frac{1}{2}}_{h,n,\ell,m}\ \bar{v}^{k+1}_{n,\ell,m+h})^2\Delta\xi - c\Delta t\ \|\underline{\bar{V}}^{k+1}_{\ell}\|^2, \tag{5.18}$$

where $\|\underline{\bar{V}}_\ell\|^2 = \sum_{m=0}^{M} |\bar{v}_{\ell,m}|^2\Delta\xi$, $|Y|^2 = \sum_{\ell=0}^{L} |y_\ell|^2$ and c is a constant.

Therefore, when (3.9) holds, we have

$$\bar{T}^{k+3/2}_{n,\ell} \geq -c|\bar{v}^{k+3/2}_{n,\ell,0}|^2\Delta\xi\delta_0(\lambda_{n,\ell,0}) - c|\bar{v}^{k+3/2}_{n,\ell,M}|^2\Delta\xi\delta_1(\lambda_{n,\ell,M}) - c\Delta t\|\ \underline{\bar{V}}^{k+3/2}_{\ell}\|^2$$

$$- c\Delta t\ \|\ \underline{\bar{V}}^{k+1}_{\ell}\|^2 - c\Delta\xi\,|Y^{k+1}|^2 + c_2 \sum_{m\in\underline{g}(\lambda_{n,\ell})} |\bar{v}^{k+3/2}_{n,\ell,m}|^2\Delta\xi$$

$$T^{k+1}_{n,\ell} \geq -c|\bar{v}^{k+1}_{n,\ell,0}|^2\Delta\xi\delta_0(\lambda_{n,\ell,0}) - c\ |\bar{v}^{k+1}_{n,\ell,M}|^2\Delta\xi\delta_1(\lambda_{n,\ell,M}) - c\Delta t\ \|\ \underline{\bar{V}}^{k+1}_{\ell}\|^2$$

$$+ c_2 \sum_{m\in\underline{g}(\lambda_{n,\ell})} |\bar{v}^{k+1}_{n,\ell,m}|^2\ \Delta\xi.$$

From the definition of $\hat{\underline{R}}^k$, we know

$$\|\underline{\hat{BR}}^k\underline{W}^{k+1}\|^2 = \sum_{n=1}^{N}\sum_{\ell=1}^{L} (\bar{T}^{k+3/2}_{n,\ell} + T^{k+1}_{n,\ell})$$

$$+ c_3^2(|B^{k+3/2}_g \begin{pmatrix} V_b^{k+3/2} \\ Y^{k+3/2} \end{pmatrix}|^2 + |B^{k+1}_g \begin{pmatrix} V_b^{k+1} \\ Y^{k+1} \end{pmatrix}|^2)\Delta\xi$$

$$\geq \sum_{j=k+1}^{k+3/2} \{-c(\sum_{i\in I_1} |\bar{v}^j_{bi}|^2 + |Y^j|^2)\ \Delta\xi$$

$$+ c_2 \sum_{n=1}^{N}\sum_{\ell=1}^{L}\sum_{m\in\underline{g}(\lambda_{n,\ell})} |\bar{v}^j_{n,\ell,m}|^2\Delta\xi$$

$$- c\Delta t\|\ \underline{\bar{V}}^j\|^2 + c_3^2\,|B^j_g \begin{pmatrix} V_b^j \\ Y^j \end{pmatrix}|^2\Delta\xi\}$$

$$\geq \sum_{j=k+1}^{k+3/2} \{\frac{c_2}{2}(\|\ \underline{\bar{V}}^j\|^2 + |Y^j|^2\Delta\xi)$$

$$-(c + \frac{c_2}{2})(\sum_{i\in I_1} |v^j_{bi}|^2 + |Y^j|^2)\Delta\xi$$

$$+ \frac{c_2c_3^2}{2}\sum_{i\in I_0} |v^j_{bi}|^2\Delta\xi + c_3^2|B^j_g\begin{pmatrix} V_b^j \\ Y^j \end{pmatrix}|^2\Delta\xi - c\Delta t\|\underline{\bar{V}}^j\|^2\},$$

where $\|\ \underline{\bar{V}}\|^2 = \sum_{\ell=1}^{L} \|\ \underline{\bar{V}}_\ell\|^2$. Here some relations similar to (4.17) are used.

Because (3.22) holds and c_2 is positive, we can choose such a c_3 that

$$-(c + \frac{c_2}{2})(\sum_{i\in I_1} |v^j_{bi}|^2 + |Y^j|^2)$$

$$+ \frac{c_2c_3^2}{2}\sum_{i\in I_0} |v^j_{bi}|^2 + c_3^2\ |B^j_g\ \begin{pmatrix} V_b^j \\ Y^j \end{pmatrix}|^2 \geq 0,$$

$$j = k+1,\ \ k+3/2.$$

Consequently, if $c\Delta t < \frac{c_2}{4}$, then noting $c_3 > 1$, we have

$$\|\underline{\hat{BR}}^k\underline{W}^{k+1}\|^2 \geq \frac{c_2}{4}\sum_{j=k+1}^{k+3/2} (\|\ \underline{\bar{V}}^j\ \|^2 + |Y^j|^2\Delta\xi)$$

$$\geq \frac{c_2}{4}\ \|\ \underline{W}^{k+1}\|^2,$$

that is

$$\|(\underline{\hat{BR}}^k)^{-1}\|^2 \leq \frac{4}{c_2}.$$

The boundedness of $\|\underline{\hat{BR}}^k\|^2$ is obvious. Therefore we have proved (5.14) holds.

Clearly, noticing (5.15) and (5.16), we can obtain the following inequalities on $\bar{T}^{k+3/2}_{n,\ell}$ and $T^{k+1}_{n,\ell}$:

$$\bar{T}^{k+3/2}_{n,\ell} = \sum_{m\in\underline{g}(\lambda_{n,\ell})} (\sum_h(\gamma^{k+1}_{h,n,\ell,m}\bar{v}^{k+3/2}_{n,\ell,m+h} + 0(\Delta t)\ \bar{v}^{k+3/2}_{\ell,m+h} + 0(\Delta t)\bar{v}^{k+1}_{\ell,m+h}) +$$

$$+ 0^*(\Delta t)Y^{k+1})^2\Delta\xi$$

$$= \sum_{m\in\underline{g}(\lambda_{n,\ell})} (\sum_h(s^{k+1}_{h,n,\ell,m}\bar{v}^{k+\frac{1}{2}}_{n,\ell,m+h} + 0(\Delta t)\bar{v}^{k+\frac{1}{2}}_{\ell,m+h}) + 0^*(\Delta t)Y^{k+\frac{1}{2}})^2\Delta\xi \tag{5.19}$$

$$\leq \sum_{m\in\underline{g}(\lambda_{n,\ell})} (\sum_h s^{k+1}_{h,n,\ell,m}\bar{v}^{k+\frac{1}{2}}_{n,\ell,m+h})^2\Delta\xi + c\Delta t\|\ \underline{\bar{V}}^{k+\frac{1}{2}}_{\ell}\|^2 + c\Delta\xi|Y^{k+\frac{1}{2}}|^2$$

and

$$T^{k+1}_{n,\ell} = \sum_{m\in \underline{g}(\lambda_{n,\ell})} \Big(\sum_h (\gamma^{k+\frac12}_{h,n,\ell,m}\bar{v}^{k+1}_{n,\ell,m+h} + 0(\Delta t)\bar{v}^{k+1}_{\ell,m+h})\Big)^2 \Delta\xi$$

$$= \sum_{m\in \underline{g}(\lambda_{n,\ell})} \Big(\sum_h (s^{k+\frac12}_{h,n,\ell,m}\bar{v}^{k}_{n,\ell,m+h} + 0(\Delta t)\bar{v}^{k}_{\ell,m+h} + 0(\Delta t)\bar{v}^{k+\frac12}_{\ell,m+h})$$

$$+ 0^*(\Delta t)Y^k + 0^*(\Delta t)Y^{k+\frac12}\Big)^2\Delta\xi$$

$$\le \sum_{m\in g(\lambda_{n,\ell})} \Big(\sum_h (s^{k+\frac12}_{h,n,\ell,m}\bar{v}^{k}_{n,\ell,m+h})\Big)^2\Delta\xi + c\Delta t\|\underline{\bar{V}}^k_\ell\|^2 +$$

$$+ c\Delta t\|\underline{\bar{V}}^{k+\frac12}_\ell\|^2 + c\Delta\xi(|Y^k|^2 + |Y^{k+\frac12}|^2). \qquad (5.20)$$

Therefore when (3.10) holds, from (5.17)-(5.20) we have

$$\bar{T}^{k+3/2}_{n,\ell} + T^{k+1}_{n,\ell} - \bar{T}^{k+\frac12}_{n,\ell} - T^{k}_{n,\ell} \le$$

$$\le \sum_{j=k}^{k+\frac12} \Big(\sum_{m\in \underline{g}(\lambda_{n,\ell})} \big(\sum_h s^{j+\frac12}_{h,n,\ell,m}\bar{v}^j_{n,\ell,m+h}\big)^2\Delta\xi$$

$$- \sum_{m\in \underline{g}(\lambda_{n,\ell})} \big(\sum_h \gamma^{j+\frac12}_{h,n,\ell,m}\bar{v}^j_{n,\ell,m+h}\big)^2\Delta\xi$$

$$+ c\Delta t\|\underline{\bar{V}}^j_\ell\|^2 + c\Delta\xi|Y^j|^2\Big)$$

$$\le \sum_{j=k}^{k+\frac12} [(c\delta_0(\lambda_{n,\ell,0}) - c_2|\sigma^j_{n,\ell,0}|(1-\delta_0(\lambda_{n,\ell,0})))|\bar{v}^j_{n,\ell,0}|^2\Delta\xi$$

$$+ (c\delta_1(\lambda_{n,\ell,M}) - c_2|\sigma^j_{n,\ell,M}|(1-\delta_1(\lambda_{n,\ell,M})))|\bar{v}^j_{n,\ell,M}|^2\Delta\xi$$

$$+ c\Delta t\|\underline{\bar{V}}^j_\ell\|^2 + c\Delta\xi|Y^j|^2],$$

$$n = 1,2,\cdots,N, \qquad \ell = 1,2,\cdots,L.$$

Furthermore, noticing $\underline{B}^{k+1}_g\begin{pmatrix}\underline{V}^{k+1}\\ Y^{k+1}\end{pmatrix} = B^{k+1}_g\begin{pmatrix}V^{k+1}_b\\ Y^{k+1}\end{pmatrix} = 0$, we have

$$\|\underline{\hat{BR}}^k\underline{W}^{k+1}\|^2 - \|\underline{\hat{BR}}^k\underline{W}^k\|^2$$

$$= \sum_{n=1}^{N}\sum_{\ell=1}^{L}(\bar{T}^{k+3/2}_{n,\ell} + T^{k+1}_{n,\ell} - \bar{T}^{k+\frac12}_{n,\ell} - T^k_{n,\ell})$$

$$+ c_3^2\Big(\Big|B^{k+3/2}_g\begin{pmatrix}V^{k+3/2}_b\\ Y^{k+3/2}\end{pmatrix}\Big|^2 + \Big|B^{k+1}_g\begin{pmatrix}V^{k+1}_b\\ Y^{k+1}\end{pmatrix}\Big|^2$$

$$- \Big|B^{k+\frac12}_g\begin{pmatrix}V^{k+\frac12}_b\\ Y^{k+\frac12}\end{pmatrix}\Big|^2 - \Big|B^k_g\begin{pmatrix}V^k_b\\ Y^k\end{pmatrix}\Big|^2\Big)\Delta\xi$$

$$\le \sum_{j=k}^{k+\frac12}\Big(\Big(c\sum_{i\in I_1}|\bar{v}^j_{bi}|^2 - c_2\frac{\Delta t}{\Delta\xi}\sum_{i\in I_0}|\lambda^j_{bi}||\bar{v}^j_{bi}|^2 + c|Y^j|^2\Big)\Delta\xi$$

$$+ c\Delta t\|\underline{\bar{V}}^j\|^2\Big)$$

$$= \sum_{j=k}^{k+\frac12}\Big(\Big(c\sum_{i\in I_1}|v^j_{bi}|^2 - c_2c_3^2\frac{\Delta t}{\Delta\xi}\sum_{i\in I_0}|\lambda^j_{bi}||v^j_{bi}|^2 + c|Y^j|^2\Big)\Delta\xi$$

$$+ c\Delta t\|\underline{\bar{V}}^j\|^2\Big).$$

When (3.20) holds and E = 0 we can choose such a constant c_3 that

$$c|Y^j|^2 + c\sum_{i\in I_1}|v^j_{bi}|^2 - c_2c_3^2\frac{\Delta t}{\Delta\xi}\sum_{i\in I_0}|\lambda^j_{bi}||v^j_{bi}|^2 \le 0.$$

Consequently, we have derived

$$\|\underline{\hat{BR}}^k\underline{W}^{k+1}\|^2 - \|\underline{\hat{BR}}^k\underline{W}^k\|^2$$

$$\le c\Delta t(\|\underline{\bar{V}}^k\|^2 + \|\underline{\bar{V}}^{k+\frac12}\|^2)$$

$$\le cc_3^2\,\Delta t\,\|\underline{W}^k\|^2,$$

that is, we have proved that (5.13) holds. (5.13) and (5.14) hold, so (5.12) is stable. Consequently, (5.11) holds, which means $\begin{pmatrix}\underline{U}\\ Z\end{pmatrix}$ converges to $\begin{pmatrix}\tilde{\underline{U}}\\ \tilde{Z}\end{pmatrix}$ with a convergence rate of order Δt^2.

Finally, we would like to point out that for scheme (2.16), the same result can be obtained. That is, if the scheme (2.16) has Property A, the boundary condition has Property B, and the solution in every subregion has certain smoothness, then there exists a solution $\begin{pmatrix}\underline{U}^{k+1}\\ Z^{k+1}\end{pmatrix}$ of (2.16) which satisfies

$$\Big\|\begin{pmatrix}\underline{U}^{k+1} - \tilde{\underline{U}}^{k+1}\\ Z^{k+1} - \tilde{Z}^{k+1}\end{pmatrix}\Big\| \le c\Delta t^2, \quad \text{for} \quad (k+1)\Delta t \le T.$$

6. SOME DISCUSSION ON PROPERTY A

In Section 3, we point out that schemes (2.5) and (3.14) possess Property A for the case of constant coefficients. Indeed, this fact is still true for the case of variable coefficients and many schemes are of Property A. In this section, we shall prove schemes (2.5) and (2.6) possess Property A for the case of variable coefficients.

First we give two Lemmas

Lemma 1 If $\sum_\ell (-1)^{H_3(\ell)} d^*_\ell(\theta)d_\ell(\theta) \equiv 0$, then the matrix $Q = \sum_\ell (-1)^{H_3(\ell)} D^*_\ell D_\ell$ is a pseudo-null matrix, i.e., the sums of the elements on every diagonal line of the matrix are all equal to zero, where $H_3(\ell)$ is equal to either 0 or 1,

$$d_\ell(\theta)=\sum_{h=H_1}^{H_2} d_{\ell,h}e^{ih\theta},\quad D_\ell=(d_{\ell,H_1},\ d_{\ell,H_1+1},\dots,d_{\ell,H_2})$$

and the symbol "*" represents conjugate transposition for vectors and conjugation for scalar quantities.

Proof. Because of

$$\sum_\ell (-1)^{H_3(\ell)}\Big(\sum_{h=H_1}^{H_2} d_{\ell,h}e^{ih\theta}\Big)^*\Big(\sum_{h=H_1}^{H_2} d_{\ell,h}e^{ih\theta}\Big)$$

$$=\sum_\ell(-1)^{H_3(\ell)}\sum_{h=-H}^{H}\Big(\sum_j d^*_{\ell,j}\,d_{\ell,j+h}\Big)e^{ih\theta}$$

$$=\sum_{h=-H}^{H}\Big(\sum_\ell(-1)^{H_3(\ell)}\sum_j d^*_{\ell,j}\,d_{\ell,j+h}\Big)e^{ih\theta}\equiv 0,$$

for any h, we have

$$\sum_\ell (-1)^{H_3(\ell)}\sum_j d^*_{\ell,j}\,d_{\ell,j+h}=0,$$

where $H=H_2-H_1$, and j in summation formula runs over all the values satisfying both $H_1\le j+h\le H_2$ and $H_1\le j\le H_2$.

On the other hand

$$Q=\sum_\ell(-1)^{H_3(\ell)}D^*_\ell D_\ell$$

$$=\sum_\ell(-1)^{H_3(\ell)}\begin{bmatrix} d^*_{\ell,H_1}d_{\ell,H_1}, & d^*_{\ell,H_1}d_{\ell,H_1+1}, & \dots, & d^*_{\ell,H_1}d_{\ell,H_2}\\ d^*_{\ell,H_1+1}d_{\ell,H_1}, & d^*_{\ell,H_1+1}d_{\ell,H_1+1}, & \dots, & d^*_{\ell,H_1+1}d_{\ell,H_2}\\ \cdots & \cdots & \cdots & \cdots\\ d^*_{\ell,H_2}d_{\ell,H_1}, & d^*_{\ell,H_2}d_{\ell,H_1+1}, & \dots, & d^*_{\ell,H_2}d_{\ell,H_2}\end{bmatrix}$$

and the sum of elements on the h-th diagonal line just is $\sum_\ell(-1)^{H_3(\ell)}\sum_j d^*_{\ell,j}d_{\ell,j+h}$, where the h-th diagonal line denotes the main diagonal line if h=0, it denotes the h-th upper-diagonal line if $h > 0$, and it represents the $|h|$-th lower-diagonal line if $h < 0$. Therefore, the conclusion is proved.

Lemma 2 If $d_1^*(\theta)\,d_1(\theta)-d_2^*(\theta)\,d_2(\theta)\ge 0$, then the matrix $D_1^*D_1-D_2^*D_2$ can be represented by a sum of one nonnegative definite matrix Z and one pseudo-null matrix Q.

Proof. Because of $d_1^*(\theta)d_1(\theta)-d_2^*(\theta)d_2(\theta)\ge 0$, one can find a $c(\theta)=\sum_{h=H_1}^{H_2}c_h e^{ih\theta}$ such that $d_1^*(\theta)d_1(\theta)-d_2^*(\theta)d_2(\theta)\equiv c^*(\theta)c(\theta)$ (The Fejér-Riesz Theorem).

Hence, we know from Lemma 1 that $Q=D_1^*D_1-D_2^*D_2-C^*C$ is a pseudo-null matrix, where $C=(c_{H_1},c_{H_1+1},\dots,c_{H_2})$. Obviously C^*C is a nonnegative definite matrix. Therefore, the conclusion of this lemma is true.

From Lemma 2, it can be easily seen that the conditions (3.3) and (3.4) are closely related to (3.10) and (3.9) respectively.

In fact, according to the definition of T^k and noting (3.2), we have

$$\begin{cases} T^{k+1}=\displaystyle\sum_{m\in \underline{g}(\lambda)}\Big(\sum_{h=H_1}^{H_2}\gamma^{k+\frac12}_{h,m}u^{k+1}_{m+h}\Big)^2\Delta\xi\\ \quad=\displaystyle\sum_{m\in \underline{g}(\lambda)}\Big(\sum_{h=H_1}^{H_2}s^{k+\frac12}_{h,m}u^{k}_{m+h}\Big)^2\Delta\xi\\ \quad=\displaystyle\sum_{m\in \underline{g}(\lambda)}(S^*SU,\ U)^k_m\,\Delta\xi+O(\Delta\xi)\,\|u^k\|^2\\ T^{k}=\displaystyle\sum_{m\in \underline{g}(\lambda)}\Big(\sum_{h}\gamma^{k-\frac12}_{h,m}u^{k}_{m+h}\Big)^2\Delta\xi\\ \quad=\displaystyle\sum_{m\in \underline{g}(\lambda)}(R^*RU,\ U)^k_m\,\Delta\xi+O(\Delta\xi)\,\|u^k\|^2. \end{cases}\tag{6.1}$$

When deducing (6.1), we use the conditions (3.5) and (3.6), assume $\Delta t/\Delta\xi$ to be bounded and adopt the following symbols

$$\begin{cases} R^k_m=(\gamma^k_{H_1,m},\ \gamma^k_{H_1+1,m},\dots,\gamma^k_{H_2,m})\big|_{\Delta\xi=\Delta t=0},\\ S^k_m=(s^k_{H_1,m},\ s^k_{H_1+1,m},\dots,s^k_{H_2,m})\big|_{\Delta\xi=\Delta t=0},\\ (AU,U)^k_m=\displaystyle\sum_{i=1}^{H+1}\sum_{j=1}^{H+1}(a_{i,j}u^k_{m+H_1-1+i}u^k_{m+H_1-1+j})\\ a_{i,j}\ \text{being the element located on the i-th row and the j-th column of A.} \end{cases}$$

From Lemma 2, it follows that if the conditions (3.4) and (3.3) are satisfied, then there exist certain nonnegative definite matrices $\bar Z$, Z and certain pseudo-null matrices $\bar Q$, Q such that

$$(R^*RU,\ U)_m-c_1|u_m|^2=(\bar Z\,U,\ U)_m+(\bar Q\,U,\ U)_m,$$

$$((R^*R-S^*S)U,U)_m=(Z\,U,U)_m+(Q\,U,\ U)_m\,.$$

Therefore, we have

$$T^k\ge\Big[c_1\sum_{m\in \underline{g}(\lambda)}|u^k_m|^2+\sum_{m\in \underline{g}(\lambda)}(\bar Q\,U,U)^k_m\Big]\Delta\xi-O(\Delta\xi)\,\|u^k\|^2\tag{6.2}$$

and

$$T^{k+1}-T^{k} \le -\sum_{m\in \underline{g}(\lambda)} (Q\,U,U)^{k}_{m}\Delta\xi + O(\Delta\xi)\,\|u^{k}\|^{2} . \qquad (6.3)$$

From the property of pseudo-null matrices, we know that if every element of $\bar{Q}$ and Q satisfies the Lipschitz condition as ξ varies, then in the above inequality

$$\sum_{m\in \underline{g}(\lambda)} (\bar{Q}\,U,\,U)^{k}_{m} \quad \text{and} \quad -\sum_{m\in \underline{g}(\Sigma)} (Q\,U,\,U)^{k}_{m}$$

may be replaced by sums of $O(\Delta\xi)\sum_{m=0}^{M} |u^{k}_{m}|^{2}$ and certain quadratic forms of u^{k}_{m} on points near the boundaries. Therefore, (6.2) and (6.3) are respectively similar to (3.9) and (3.10). Consequently, it is quite common for a scheme to possess Property A.

We now prove that scheme (2.5) possesses Property A for the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \lambda(\xi,t)\,\dfrac{\partial u}{\partial \xi} = 0, \\ \lambda(\xi,t) \ge \varepsilon > 0,\ 0 \le \xi \le 1. \end{cases} \qquad (6.4)$$

For (6.4), scheme (2.5) is in the form

$$\gamma^{k+\frac{1}{2}}_{-1,m}u^{k+1}_{m-1}+\gamma^{k+\frac{1}{2}}_{0,m}u^{k+1}_{m} = s^{k+\frac{1}{2}}_{-1,m}u^{k}_{m-1}+ s^{k+\frac{1}{2}}_{0,m}u^{k}_{m}, \qquad m = 1,2,\cdots,M, \qquad (6.5)$$

where

$$\gamma^{k+\frac{1}{2}}_{-1,m} = \frac{1}{2} - \frac{1}{2}\mu\sigma^{k+\frac{1}{2}}_{m-\frac{1}{2}} ,$$

$$\gamma^{k+\frac{1}{2}}_{0,m} = \frac{1}{2} + \frac{1}{2}\mu\sigma^{k+\frac{1}{2}}_{m-\frac{1}{2}} ,$$

$$s^{k+\frac{1}{2}}_{-1,m} = \frac{1}{2} + \frac{1}{2}\mu\sigma^{k+\frac{1}{2}}_{m-\frac{1}{2}} ,$$

$$s^{k+\frac{1}{2}}_{0,m} = \frac{1}{2} - \frac{1}{2}\mu\sigma^{k+\frac{1}{2}}_{m-\frac{1}{2}} .$$

For this type of scheme, the condition (3.4) is in the form

$$\gamma^{*}(\theta)\gamma(\theta) = 1 - 4\gamma_{0}(1-\gamma_{0})\sin^{2}\frac{\theta}{2} \ge c_{1} > 0, \qquad (6.6)$$

where γ_0 denotes $\gamma_0(0,\,0)$. Moreover $\gamma^{*}(\theta)\gamma(\theta)$ can be further rewritten as

$$\gamma^{*}(\theta)\gamma(\theta) = c_1' + (a + be^{i\theta})^{*}\,(a + be^{i\theta}),$$

where

$$c_1' = \min\left\{ \frac{1}{2},\ \frac{c_1}{2} \right\},$$

$$a = \frac{\sqrt{1-c_1'} - \sqrt{(2\gamma_0-1)^{2} - c_1'}}{2},$$

$$b = \frac{\sqrt{1-c_1'} + \sqrt{(2\gamma_0-1)^{2} - c_1'}}{2}.$$

Consequently, according to Lemma 1, the matrix

$$\bar{Q}_2 \equiv R^{*}R - \begin{pmatrix} 0 & 0 \\ 0 & c_1' \end{pmatrix} - Z_2$$

is a pseudo-null matrix, where $Z_2 = \binom{a}{b}\,(a\ \ b)$. Actually,

$$\bar{Q}_2 = \begin{bmatrix} (1-\gamma_0)^{2} - a^{2} & 0 \\ 0 & \gamma_0^{2} - c_1' - b^{2} \end{bmatrix} .$$

Moreover, when (6.6) holds, we have $(2\gamma_0-1)^{2} - c_1' \ge \frac{c_1}{2} > 0$, which guarantees that every element of $\bar{Q}_2$ satisfies the Lipschitz condition with respect to ξ. Therefore, since Z_2 is nonnegative definite, we have the following inequality

$$\begin{aligned} T=\sum_{m=1}^{M}(R^{*}RU,U)_{m}\Delta\xi + O(\Delta\xi)\|u\|^{2} &\ge c_1'\sum_{m=1}^{M}|u_m|^{2}\Delta\xi + \sum_{m=1}^{M}(\bar{Q}_2U,U)_{m}\Delta\xi - O(\Delta\xi)\|u\|^{2} \\ &\ge c_1'\sum_{m=1}^{M}|u_m|^{2}\Delta\xi + ((1-\gamma_0)^{2}-a^{2})_1\,|u_0|^{2}\Delta\xi \\ &\quad + (\gamma_0^{2} - c_1' - b^{2})_M\,|u_M|^{2}\Delta\xi - O(\Delta\xi)\,\|u\|^{2} \\ &\ge c_2\sum_{m=1}^{M}|u_m|^{2}\Delta\xi - c|u_0|^{2}\Delta\xi - O(\Delta\xi)\,\|u\|^{2}, \end{aligned} \qquad (6.7)$$

where $c_2 = \min\{ c_1',\ (\gamma_0^{2} - b^{2})_M \}$, $\quad c > |((1-\gamma_0)^{2} - a^{2})_1|$.

Noting

$$\frac{\partial b^{2}}{\partial c_1'} = 2b\,\frac{1}{4}\left[\frac{-1}{\sqrt{1-c_1'}} + \frac{-1}{\sqrt{(2\gamma_0-1)^{2} - c_1'}} \right] = \frac{-b^{2}}{\sqrt{1-c_1'}\ \sqrt{(2\gamma_0-1)^{2}-c_1'}} < 0 ,$$

$$(\gamma_0^{2} - b^{2})\Big|_{c_1' = 0} = 0,$$

and $c_1' > 0$, we know $(\gamma_0^{2} - b^{2})_M > 0$. Therefore (6.7) can be written in form (3.9).

For scheme (6.5), the condition (3.3) is

$$\gamma^{*}(\theta)\gamma(\theta) - s^{*}(\theta)s(\theta) = 4(s_0(1-s_0) - \gamma_0(1-\gamma_0))\sin^{2}\frac{\theta}{2} \ge 0, \qquad (6.8)$$

where s_0 denotes $s_0(0,0)$. Since $4\sin^{2}\frac{\theta}{2} = (1-e^{i\theta})^{*}\,(1-e^{i\theta})$, according to Lemma 1, the matrix

$$Q_2 \equiv \begin{bmatrix} (1-\gamma_0)^{2} & \gamma_0(1-\gamma_0) \\ \gamma_0(1-\gamma_0)^{2} & \gamma_0^{2} \end{bmatrix} - \begin{bmatrix} (1-s_0)^{2} & s_0(1-s_0) \\ s_0(1-s_0) & s_0^{2} \end{bmatrix}$$

$$-(s_0(1-s_0)-\gamma_0(1-\gamma_0))\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$$

is a pseudo-null matrix. Actually

$$Q_2=\begin{bmatrix} s_0-\gamma_0 & 0\\ 0 & \gamma_0-s_0\end{bmatrix}$$

Clearly, if (6.8) is valid, $(s_0(1-s_0)-\gamma_0(1-\gamma_0))\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$ is nonnegative definite. Therefore, we have

$$\sum_{m=1}^{M}((S^*S-R^*R)U,U)_m\Delta\xi \le -\sum_{m=1}^{M}(Q_2U,U)_m\Delta\xi$$

$$\le(\gamma_0-s_0)_0|u_0|^2\Delta\xi-(\gamma_0-s_0)_M|u_M|^2\Delta\xi+O(\Delta\xi)\,\|u\|^2,$$

from which and noticing $\gamma_0-s_0=\mu\sigma$, we can obtain the following inequality in form (3.10):

$$T^{k+1}-T^k\le\sigma_0|u_0^k|^2\Delta\xi-\sigma_M|u_M^k|^2\Delta\xi+O(\Delta\xi)\,\|u^k\|^2.$$

In what follows, we prove that scheme (2.6) possesses Property A when it is applied to

$$\begin{cases}\dfrac{\partial u}{\partial t}+\lambda(\xi,t)\dfrac{\partial u}{\partial\xi}=0,\\ \lambda(0,t)\ge 0,\quad \lambda(1,t)\le 0, \qquad 0\le\xi\le 1.\end{cases}\tag{6.9}$$

In this case, the difference equations are

$$u_m^{k+1}=s_{-1,m}^{k+\frac12}\,u_{m-1}^k+s_{0,m}^{k+\frac12}\,u_m^k+s_{1,m}^{k+\frac12}\,u_{m+1}^k,\qquad m=1,2,\cdots,M-1,\tag{6.10}$$

where

$$s_{-1}=\frac12(1+\sigma)\sigma,\quad s_0=(1+\sigma)(1-\sigma),\quad s_1=-\frac12(1-\sigma)\sigma.$$

For explicit schemes, (3.9) always holds. Therefore we only need to prove (3.10) is satisfied. Now condition (3.3) is

$$\gamma^*(\theta)\gamma(\theta)-s^*(\theta)s(\theta)=4(1-\sigma^2)\sigma^2\sin^4\frac{\theta}{2}\ge 0.\tag{6.11}$$

Since $\sin^4\frac{\theta}{2}=\frac{1}{16}(e^{-i\theta}-2+e^{i\theta})^*(e^{-i\theta}-2+e^{i\theta})$, according to Lemma 1, the matrix

$$Q_3\equiv\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}-\begin{bmatrix}s_{-1}^2 & s_{-1}s_0 & s_{-1}s_1\\ s_0s_{-1} & s_0^2 & s_0s_1\\ s_1s_{-1} & s_1s_0 & s_1^2\end{bmatrix}-Z_3$$

$$=\begin{bmatrix}-s_{-1}^2-a & -s_{-1}s_0+2a & 0\\ -s_0s_1+2a & 1-s_0^2-4a & -s_0s_1+2a\\ 0 & -s_1s_0+2a & -s_1^2-a\end{bmatrix}$$

is a pseudo-null matrix, where

$$Z_3=a\begin{bmatrix}1\\-2\\1\end{bmatrix}(1\quad -2\quad 1),$$

$$a=\frac14(1-\sigma^2)\sigma^2.$$

Because Z_3 is a nonnegative definite matrix when (6.11) holds and because Q_3 is Lipschitz continuous with respect to ξ, we have

$$\sum_{m=1}^{M-1}((S^*S-R^*R)U,U)_m\Delta\xi\le-\sum_{m=1}^{M-1}(Q_3U,U)_m\Delta\xi$$

$$\le(Q_uU,U)_0\,\Delta\xi+(Q_\ell U,U)_M\,\Delta\xi+O(\Delta\xi)\,\|u\|^2,\tag{6.12}$$

where

$$(Q_uU,U)_0=(u_0,u_1)\begin{bmatrix}s_{-1}^2+a & s_{-1}s_0-2a\\ s_0s_{-1}-2a & s_{-1}^2+s_0^2-1+5a\end{bmatrix}_0\begin{pmatrix}u_0\\u_1\end{pmatrix},$$

$$(Q_\ell U,U)_M=(u_{M-1},u_M)\begin{bmatrix}s_1^2+s_0^2-1+5a & s_{-1}s_0-2a\\ s_{-1}s_0-2a & s_1^2+a\end{bmatrix}_M\begin{pmatrix}u_{M-1}\\u_M\end{pmatrix}.$$

Since

$$\begin{bmatrix}s_{-1}^2+a & s_{-1}s_0-2a\\ s_0s_{-1}-2a & s_{-1}^2+s_0^2-1+5a\end{bmatrix}_0=\frac12\begin{bmatrix}(1+\sigma)\sigma^2 & (1-\sigma^2)\sigma\\ (1-\sigma^2)\sigma & \sigma^2(\sigma-1)\end{bmatrix}_0$$

$$=\frac12\begin{bmatrix}(1+\sigma)^2(\sigma-1) & (1-\sigma^2)\sigma\\ (1-\sigma^2)\sigma & \sigma^2(\sigma-1)\end{bmatrix}_0+\begin{bmatrix}\frac12(1+\sigma) & 0\\ 0 & 0\end{bmatrix}_0$$

and the first matrix on the right hand side of the last sign of equality is nonpositive definite, it follows that

$$(Q_uU,U)_0\le\frac12(1+\sigma_0)\,|u_0|^2.$$

Similarly, we can obtain

$$(Q_\ell U,U)_M\le\frac12(1+\sigma_M)\,|u_M|^2.$$

Therefore from (6.12) we can get (3.10), which means that scheme (2.6) possesses Property A when it is applied to (6.9).

Indeed, many schemes have Property A. For more results, the reader is referred to book [3].

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LECTURE NOTES IN MATHEMATICS
Edited by A. Dold and B. Eckmann

Some general remarks on the publication of proceedings of congresses and symposia

Lecture Notes aim to report new developments - quickly, informally and at a high level. The following describes criteria and procedures which apply to proceedings volumes.

1. One (or more) expert participant(s) of the meeting should act as the responsible editor(s) of the proceedings. They select the papers which are suitable (cf. points 2, 3) for inclusion in the proceedings, and have them individually refereed (as for a journal). It should not be assumed that the published proceedings must reflect conference events faithfully and in their entirety. Contributions to the meeting which are not included in the proceedings can be listed by title. The series editors will normally not interfere with the editing of a particular proceedings volume - except in fairly obvious cases, or on technical matters, such as described in points 2, 3. The names of the responsible editors appear on the title page of the volume.

2. The proceedings should be reasonably homogeneous (concerned with a limited area). For instance, the proceedings of a congress on "Analysis" or "Mathematics in Wonderland" would normally not be sufficiently homogeneous.

   One or two longer survey articles on recent developments in the field are often very useful additions to such proceedings - even if they do not correspond to actual lectures at the congress. An extensive introduction on the subject of the congress would be desirable.

3. The contributions should be of a high mathematical standard and of current interest. Research articles should present new material and not duplicate other papers already published or due to be published. They should contain sufficient information and motivation and they should present proofs, or at least outlines of such, in sufficient detail to enable an expert to complete them. Thus resumes and mere announcements of papers appearing elsewhere cannot be included, although more detailed versions of a contribution may well be published in other places later.

   Surveys, if included, should cover a sufficiently broad topic, and should in general not simply review the author's own recent research. In the case of surveys, exceptionally, proofs of results may not be necessary.

   The editors of a volume are strongly advised to inform contributors about these points at an early stage.