time interval $[t, t+dt]$. This equals the right-hand side of the equation. Its expectation is $\sum_{i=1}^{n} \frac{\partial V}{\partial S_i} b_i \lambda_i dt$ because $\mathrm{E}[dX_i] = 0$, $i = 1, 2, \cdots, n$. Therefore, the term $\frac{\partial V}{\partial S_i} b_i \lambda_i dt$ may be interpreted as an excess return above the risk-free return for taking the risk dX_i. Consequently, λ_i is a price of risk for S_i that is associated with dX_i and is often called the market price of risk for $S_{i.}$

2.9.4 Three Types of State Variables

There are three types of state variables, for which the term $a_i - \lambda_i b_i$ in (2.81) will be determined in different ways.

Suppose S_i is an asset price that can be traded on the market. For example, S_i is a stock price per share. In this case, the stock itself can be considered as a derivative security. Suppose the stock pays a dividend continuously with dividend yield D_{0i}. In this case, the price of this derivative security should be $S_i \mathrm{e}^{-D_{0i}(T-t)}$ (see Problem 9), i.e., $S_i \mathrm{e}^{-D_{0i}(T-t)}$ should be a solution of the equation (2.81). Substituting $V = S_i \mathrm{e}^{-D_{0i}(T-t)}$ into (2.81) yields $\mathrm{e}^{-D_0(T-t)}(D_{0i}S_i + a_i - \lambda_i b_i - rS_i) = 0$. Therefore, for this case

$$a_i - \lambda_i b_i = (r - D_{0i})S_i. \tag{2.82}$$

We obtain the same result as we had when the Black–Scholes equation for continuous dividend-paying assets was derived. If the dividend is paid discretely, the situation is similar:

$$a_i - \lambda_i b_i = rS_i - D_i(S_i, t) \tag{2.83}$$

because if V depends only on S_i and t, then (2.81) should become (2.13). Here, $D_i(S_i, t)dt$ is the dividend paid during the time period $[t, t+dt]$.

A state variable S_i with $b_i = 0$ in (2.76) is another type of state variable. From $b_i = 0$, we have

$$a_i - \lambda_i b_i = a_i, \tag{2.84}$$

so λ_i disappears in the equation (2.81). As we will see from Chapter 3, if S_i is the maximum, minimum, or average price of the stock during a time period, then $dS_i = a_i dt$.

If S_i is the spot interest rate, then in order to determine λ_i, we have to solve an inverse problem. We will discuss this problem in detail in Chapter 4. This is an example of the third type of state variable. Besides the interest rate, the random volatility also falls into this type of state variable.

2.9.5 Uniqueness of Solutions

The equation (2.81) is a parabolic equation. If $b_i = 0$ at $S_i = S_{i,l}$ and $S_i = S_{i,u}$, $i = 1, 2, \cdots, n$, then we say that the equation is a degenerate parabolic partial differential equation. In this subsection, we are going to discuss when a degenerate equation has a unique solution. The conclusion expected is that if for any i,

$$a_i\left(S_{i,l}, t\right) - b_i(S_{i,l}, t)\frac{\partial}{\partial S_i}b_i(S_{i,l}, t) \geq 0 \tag{2.85}$$

and

$$a_i\left(S_{i,u}, t\right) - b_i(S_{i,u}, t)\frac{\partial}{\partial S_i}b_i(S_{i,u}, t) \leq 0 \tag{2.86}$$

hold,[14] the solution of the degenerate parabolic equation on a rectangular domain with a final condition at $t = T$ is unique.[15] If

$$a_i\left(S_{i,l}, t\right) - b_i(S_{i,l}, t)\frac{\partial}{\partial S_i}b_i(S_{i,l}, t) < 0$$

or

$$a_i\left(S_{i,u}, t\right) - b_i(S_{i,u}, t)\frac{\partial}{\partial S_i}b_i(S_{i,u}, t) > 0,$$

then a boundary condition at $S_i = S_{i,l}$ or $S_i = S_{i,u}$ needs to be imposed besides the final condition in order to have a unique solution. We now prove this conclusion for the one-dimensional case.

In the case $n = 1$, (2.81) simplifies to

$$\frac{\partial V}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial S^2} + (a - \lambda b)\frac{\partial V}{\partial S} - rV + K = 0.$$

Here, the sign of the coefficient of the second derivative is opposite of the coefficient of the second derivative in the heat equation. We say that such a parabolic equation has an "anti-directional" time. For a heat equation, an initial condition is given at $t = 0$, and the solution for $t \geq 0$ needs to be determined. Therefore, for the equation with an "anti-directional" time, a final condition should be given at $t = T$, and the solution for $t \leq T$ is needed to be determined. Consequently, we consider the following problem:

[14] a_i and b_i could also depend on $S_1, \cdots, S_{i-1}, S_{i+1}, \cdots, S_n$. Here, the dependence of a_i and b_i on them is suppressed, and the two relations hold for $S_j \in [S_{j,l}, S_{j,u}]$, $j = 1, \cdots, i-1, i+1, \cdots, n$.

[15] For a parabolic equation defined on a general domain, the conditions for a parabolic partial differential equation to be degenerate and the conditions for the solution of its initial-value problem to be unique, see the paper [81] by Zhu.

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}b^2\dfrac{\partial^2 V}{\partial S^2} + (a - \lambda b)\dfrac{\partial V}{\partial S} - rV + K = 0, \\ \qquad\qquad\qquad\qquad 0 \le t \le T, \quad S_l \le S \le S_u, \\ V(S,T) = f(S), \qquad\qquad S_l \le S \le S_u, \\ V(S_l,t) \begin{cases} \text{needs not to be given if (2.85) holds,} \\ = f_l(t) \text{ if (2.85) does not hold,} \end{cases} \\ V(S_u,t) \begin{cases} \text{needs not to be given if (2.86) holds,} \\ = f_u(t) \text{ if (2.86) does not hold.} \end{cases} \end{cases} \tag{2.87}$$

It is not difficult to convert (2.87) into a problem in the form:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = f_1(x,\tau)\dfrac{\partial^2 u}{\partial x^2} + f_2(x,\tau)\dfrac{\partial u}{\partial x} + f_3(x,\tau)u + g(x,\tau), \\ \qquad\qquad\qquad\qquad 0 \le x \le 1, \quad 0 \le \tau \le T, \\ u(x,0) = f(x), \qquad\qquad 0 \le x \le 1, \\ u(0,\tau) \begin{cases} \text{needs not to be given if } f_2(0,\tau) - \dfrac{\partial f_1(0,\tau)}{\partial x} \ge 0, \\ = f_l(\tau) \text{ if } f_2(0,\tau) - \dfrac{\partial f_1(0,\tau)}{\partial x} < 0, \end{cases} \\ u(1,\tau) \begin{cases} \text{needs not to be given if } f_2(1,\tau) - \dfrac{\partial f_1(1,\tau)}{\partial x} \le 0, \\ = f_u(\tau) \text{ if } f_2(1,\tau) - \dfrac{\partial f_1(1,\tau)}{\partial x} > 0, \end{cases} \end{cases} \tag{2.88}$$

where $f_1(0,\tau) = f_1(1,\tau) = 0$ and $f_1(x,\tau) \ge 0$. Thus, if we can prove the uniqueness of the solution of (2.88), then we have the uniqueness of the solution of (2.87). The third and fourth relations in (2.88) are the boundary conditions for degenerate parabolic equations. For parabolic equations, there is always a boundary condition at any boundary, that is, the number of boundary conditions for parabolic equations is always one. However, for degenerate parabolic equations, sometimes there is a boundary condition and sometimes there is not, depending on the value of $f_2(x,\tau) - \dfrac{\partial f_1(x,\tau)}{\partial x}$ at the boundary. For (2.88), we have the following theorem (see [68]).

Theorem 2.2 *Suppose that the solution of (2.88) exists and is bounded*[16] *and that there exist a constant c_1 and two bounded functions $c_2(\tau)$ and $c_3(\tau)$ such that*

[16] This is proven in the paper [6] by Behboudi.

$$1+\max_{0\le x\le 1,\ 0\le \tau\le T}\left(\left|\frac{\partial^2 f_1(x,\tau)}{\partial x^2}-\frac{\partial f_2(x,\tau)}{\partial x}+2f_3(x,\tau)\right|\right)\le c_1,$$

$$-\min\left(0, f_2(0,\tau)-\frac{\partial f_1(0,\tau)}{\partial x}\right)\le c_2(\tau)$$

and

$$\max\left(0, f_2(1,\tau)-\frac{\partial f_1(1,\tau)}{\partial x}\right)\le c_3(\tau).$$

In this case, its solution is unique and stable with respect to the initial value $f(x)$, inhomogeneous term $g(x,\tau)$, and the boundary values $f_l(\tau), f_u(\tau)$ if there are any.

Proof. Because the partial differential equation in (2.88) can be rewritten as

$$\frac{\partial u}{\partial \tau}=\frac{\partial}{\partial x}\left[f_1(x,\tau)\frac{\partial u}{\partial x}\right]+\left[f_2(x,\tau)-\frac{\partial f_1(x,\tau)}{\partial x}\right]\frac{\partial u}{\partial x}+f_3(x,\tau)u+g(x,\tau),$$

multiplying that equation by $2u$, we have

$$\begin{aligned}\frac{\partial (u^2)}{\partial \tau}&=2\frac{\partial}{\partial x}\left(f_1 u\frac{\partial u}{\partial x}\right)+\left(f_2-\frac{\partial f_1}{\partial x}\right)\frac{\partial (u^2)}{\partial x}-2f_1\left(\frac{\partial u}{\partial x}\right)^2+2f_3u^2+2gu\\&=2\frac{\partial}{\partial x}\left(f_1 u\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left[\left(f_2-\frac{\partial f_1}{\partial x}\right)u^2\right]-2f_1\left(\frac{\partial u}{\partial x}\right)^2\\&\quad+\left(\frac{\partial^2 f_1}{\partial x^2}-\frac{\partial f_2}{\partial x}+2f_3\right)u^2+2gu.\end{aligned}$$

Integrating this equality with respect to x on the interval $[0,1]$, we obtain the second equality

$$\begin{aligned}&\frac{d}{d\tau}\int_0^1 u^2(x,\tau)dx\\&=2\left(f_1 u\frac{\partial u}{\partial x}\right)\bigg|_{x=0}^1+\left[\left(f_2-\frac{\partial f_1}{\partial x}\right)u^2\right]\bigg|_{x=0}^1-2\int_0^1 f_1\left(\frac{\partial u}{\partial x}\right)^2dx\\&\quad+\int_0^1\left(\frac{\partial^2 f_1}{\partial x^2}-\frac{\partial f_2}{\partial x}+2f_3\right)u^2dx+2\int_0^1 gudx.\end{aligned}$$

Because

$$\begin{aligned}&\left[\left(f_2-\frac{\partial f_1}{\partial x}\right)u^2\right]\bigg|_{x=0}^1\\&=\left[f_2(1,\tau)-\frac{\partial f_1(1,\tau)}{\partial x}\right]u^2(1,\tau)-\left[f_2(0,\tau)-\frac{\partial f_1(0,\tau)}{\partial x}\right]u^2(0,\tau)\\&\le\max\left(0,f_2(1,\tau)-\frac{\partial f_1(1,\tau)}{\partial x}\right)f_u^2(\tau)-\min\left(0,f_2(0,\tau)-\frac{\partial f_1(0,\tau)}{\partial x}\right)f_l^2(\tau),\end{aligned}$$

from the equality above and the relations $f_1(0,\tau) = f_1(1,\tau) = 0$ and $f_1(x,\tau) \geq 0$, we have

$$\frac{d}{d\tau}\int_0^1 u^2(x,\tau)dx$$
$$\leq c_1\int_0^1 u^2(x,\tau)dx + \int_0^1 g^2(x,\tau)dx + c_2(\tau)f_l^2(\tau) + c_3(\tau)f_u^2(\tau).$$

Based on this inequality and by the Gronwall inequality, we arrive at

$$\int_0^1 u^2(x,\tau)dx$$
$$\leq \mathrm{e}^{c_1\tau}\left\{\int_0^1 f^2(x)dx + \int_0^\tau \left[\int_0^1 g^2(x,s)dx + c_2(s)f_l^2(s) + c_3(s)f_u^2(s)\right]ds\right\},$$
$$\tau \in [0,T].$$

From the last inequality, we know that the solution is stable with respect to $f(x)$ and $g(x,\tau)$. Also if

$$f_2(0,\tau) - \frac{\partial f_1(0,\tau)}{\partial x} \geq 0 \quad \text{and} \quad f_2(1,\tau) - \frac{\partial f_1(1,\tau)}{\partial x} \leq 0$$

hold and

$$f(x) \equiv 0, \quad g(x,\tau) \equiv 0,$$

then the solution of (2.88) must be zero. Hence, the functions $f(x)$ and $g(x,\tau)$ determine the solution uniquely. If

$$f_2(0,\tau) - \frac{\partial f_1(0,\tau)}{\partial x} < 0 \quad \text{and} \quad f_2(1,\tau) - \frac{\partial f_1(1,\tau)}{\partial x} \leq 0$$

hold, then the solution is determined by

$$f(x), \quad g(x,\tau), \quad \text{and} \quad f_l(\tau)$$

uniquely. The situation for other cases are similar. Therefore, we may conclude that if the solution of (2.88) exists, then it is unique and stable with respect to the initial value $f(x)$, the inhomogeneous term $g(x,\tau)$, and the boundary values $f_l(\tau), f_u(\tau)$ if there are any. This completes the proof and gives an explanation on when a boundary condition is necessary. ■

Here we give some remarks.

- From the probabilistic point of view, a boundary condition on a boundary is needed if and only if there are paths reaching the boundary from a point $x \in (0,1)$ and $t = 0$. Therefore, on whether or not a random variable can reach a boundary from the interior, there are similar conclusions (see [30]).

- For the case $n = 2$ and on a rectangular finite domain, if $b_1(S_1, S_2, t)$ and $b_2(S_1, S_2, t)$ are analytic, the uniqueness is also proved (see [85]) and the idea can be generalized to the case with $n > 2$. On a general finite three-dimensional domain, a similar result is also obtained (see [81]). Therefore, a degenerate parabolic equation at boundaries is similar to a hyperbolic equation.[17] Due to this fact, roughly speaking, we might say that the parabolic equation degenerates into a hyperbolic-parabolic equation (a hyperbolic equation for one-dimensional case) at the boundaries. When conditions (2.85) and (2.86) hold, incoming information is not needed at boundaries, that is, the value of V at the boundaries at $t = t^*$ is determined by the value V on the region: $S_{i,l} \le S_i \le S_{i,u}$, $i = 1, 2, \cdots, n$ and $t^* \le t \le T$. Therefore, in this case, in order for a degenerate parabolic equation to have a unique solution, only the final condition is needed.[18] If b_i is not analytic at boundaries, this conclusion has not been proved for $n > 1$. However, it is expected that the conclusion is still true.
- When the domain of S_i is not finite, a final condition is still enough for such an equation to have a unique solution if S_i can be converted into a random variable for which the reversion conditions hold. The reason is that a final condition can determine a unique solution if the new random variable is used. However, a transformation will not change the nature of the problem. If the problem has a unique solution as a function of a random variable, the problem will also have a unique solution as a function of another random variable associated by a transformation. Applying this theorem to problem (2.26), we know that its solution is unique and stable with respect to the initial value. Problem (2.26) is obtained through a transformation from the European option problem (2.24). Therefore, the European option problem (2.24) also has a unique solution.

2.10 Jump Conditions

2.10.1 Hyperbolic Equations with a Dirac Delta Function

Consider the following linear hyperbolic partial differential equation

$$\frac{\partial u}{\partial t} + f_1(x_1, x_2, \cdots, x_K, t)\frac{\partial u}{\partial x_1} + \cdots + f_K(x_1, x_2, \cdots, x_K, t)\frac{\partial u}{\partial x_K} = 0.$$

Let C be a curve defined by the system of ordinary differential equations

$$\frac{dx_1(t)}{dt} = f_1(x_1, x_2, \cdots, x_K, t),$$

[17] When $f_1(x, t) \equiv 0$, the partial differential equation in (2.88) is called a hyperbolic equation.

[18] Oleĭnik and Radkevič in their book [59] discussed the uniqueness of solutions of this type of partial differential equations under different conditions.