Math 6171 Solutions to Problems in Test 1 Fall 2001

1. (a)

$$y'' - \frac{x+6}{x^2}y' - \frac{3}{x^2}y = 0. \qquad \begin{array}{l}\text{Power series} \quad \times \\ \text{Frobenius} \quad \times\end{array}$$

(b)

$$y'' - \frac{x+2}{x+1}y' - \frac{x+3}{x+1}y = 0. \qquad \begin{array}{l}\text{Power series} \quad \surd \\ \text{Frobenius} \quad \surd\end{array}$$

(c)

$$y'' - \frac{1}{x}y' - \frac{x^2+3}{x^2}y = 0. \qquad \begin{array}{l}\text{Power series} \quad \times \\ \text{Frobenius} \quad \surd\end{array}$$

2.

$$y = \sum_{r=0}^{\infty} a_m x^{m+r}$$

$$x \cdot \sum_{m=0}^{\infty} a_m (m+r)(m+r-1) x^{m+r-2} + 2\sum_{m=0}^{\infty} a_m (m+r) x^{m+r-1} + 4x \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$$\sum_{m=0}^{\infty} a_m (m+r)(m+r+1) x^{m+r-1} + 4\sum_{m=0}^{\infty} a_m x^{m+r+1} = 0$$

$$\sum_{m=2}^{\infty} (a_m (m+r)(m+r+1) + 4a_{m-2}) x^{m+r-1} + a_0 r (r+1) x^{r-1} + a_1 (1+r)(2+r) x^r = 0$$

$$a_0 \neq 0 \Rightarrow r = 0 \text{ or } r = -1.$$

Consider

$$r = 0, \quad \Rightarrow a_1 1 \cdot 2 = 0 \Rightarrow a_1 = 0$$

$$a^m = -\frac{4}{m(m+1)} a_{m-2}, \quad m = 2, 3, \cdots, \quad \Rightarrow a_3 = a_5 = \cdots = 0$$

$$\left\{ \begin{array}{l} 0 = a_3 = a_5 = a_7 = \cdots \\ a_{2n} = \dfrac{-4}{2n(2n+1)} a_{2(n-1)} \end{array} \right., \quad n = 1, 2, \cdots$$

$$a_{2n} = \frac{-4}{2n(2n+1)} \cdot \frac{-4}{2(n-1)(2n-1)} a_{2(n-2)} = \frac{(-4)^n}{(2n+1)!} a_0$$

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_{2n} x^{2n} = a_0 \sum_{n=0}^{\infty} \frac{(-4)^n}{(2n+1)!} x^{2n} = \frac{a_0}{2x} \sum_{m=2}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x)^{2n+1} \\ &= \frac{\bar{a}_0}{x} \sin 2x. \end{aligned}$$

3.

$$\int_0^{\pi/4} \cos 8nx \sin 8mx dx = \int_0^{\pi/4} \frac{1}{2}\left[\sin(8m+8n)x + \sin(8m-8n)x\right]dx$$

$$= \begin{cases} \frac{1}{2}\left[\frac{-\cos 8(m+n)x}{8(m+n)} - \frac{\cos 8(m-n)x}{8(m-n)}\right]\Bigg|_0^{\pi/4}, & m \neq n, \\ \frac{1}{2}\frac{-\cos 8(m+n)x}{8(m+n)}\Bigg|_0^{\pi/4}, & m = n \end{cases} = 0$$

$$\int_0^{\pi/4} \cos 8nx \cos 8mx dx = \frac{1}{2}\int_0^{\pi/4} \left[\cos 8(n-m)x + \cos(8n+m)x\right]dx$$

$$= \begin{cases} \frac{1}{2}\left[\frac{\sin 8(n-m)x}{8(n-m)} + \frac{\sin 8(n+m)x}{8(n+m)}\right]\Bigg|_0^{\pi/4}, & n \neq m \\ \frac{1}{2}\left[x + \frac{\sin 8(n+m)x}{8(n+m)}\right]\Bigg|_0^{\pi/4}, & m = n \end{cases}$$

$$= \begin{cases} 0, & n \neq m \\ \frac{\pi}{8}, & n = m \end{cases}.$$

$$\int_0^{\pi/4} \sin 8nx \sin 8mx dx = \frac{1}{2}\int_0^{\pi/4} \left[\cos 8(m-n)x - \cos(8m+n)x\right]dx$$

$$= \begin{cases} \frac{1}{2}\left[\frac{\sin 8(m-n)x}{8(m-n)} - \frac{\sin 8(m+n)x}{8(m+n)}\right]\Bigg|_0^{\pi/4}, & m \neq n \\ \frac{1}{2}\left[x - \frac{\sin 8(m+n)x}{8(m+n)}\right]\Bigg|_0^{\pi/4}, & m = n \end{cases}$$

$$= \begin{cases} 0, & m \neq n \\ \frac{\pi}{8}, & m = n \end{cases}.$$

$$\int_0^{\pi/4} 1^2 dx = \frac{\pi}{4}, \quad \int_0^{\pi/4} 1 \cdot \cos 8nx dx = 0, \quad \int_0^{\pi/4} 1 \cdot \sin 8nx dx = 0,$$

Therefore, the set of functions is orthogonal and the orthonormal set is

$$\frac{1}{\sqrt{\frac{\pi}{4}}}, \quad \frac{\cos 8n\pi}{\sqrt{\frac{\pi}{8}}}, \quad \frac{\sin 8n\pi}{\sqrt{\frac{\pi}{8}}}, \quad n = 1, 2, \cdots.$$

4. (a)

$$F(s) = \frac{2}{s^2} + \frac{s-3}{(s-3)^2 + 4^2} + \frac{6}{(s-5)^2 - 6^2}$$

(b)

$$\begin{aligned} & \mathcal{L}\left\{\sin 2t \cos 3 + \sin 3 \cos 2t + e^{2+3t}\right\} \\ = \ & \cos 3 \frac{2}{s^2+4} + \sin 3 \frac{s}{s^2+4} + e^2 \frac{1}{s-3} \end{aligned}$$

(c)

$$\begin{aligned} \mathcal{L}\left\{t^2 u(t-3) + \delta(t-6)\right\} &= \mathcal{L}\left\{\left[(t-3)^2 + 6(t-3) + 9\right] u(t-3) + \delta(t-6)\right\} \\ &= \frac{2 \cdot e^{3s}}{s^3} + \frac{6e^{-3s}}{s^2} + \frac{9e^{-3s}}{s} + e^{-6s}. \end{aligned}$$

5. (a)

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+13} + \frac{1}{s^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2 + 3^2}\right\} + t \\ &= \frac{1}{3} e^{-2t} \sin 3t + t \end{aligned}$$

(b)

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2-9}\right\} = \frac{1}{3} \sinh 3(t-1)\, u(t-1)$$

(c)

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\} &= \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} dt = \int_0^t \left(e^t - 1\right) dt \\ &= e^t - t\big|_0^t = e^t - t - 1. \end{aligned}$$

6. (a)

$$\begin{cases} y_1' = 5y_1 + y_2, & y_1(0) = -3 \\ y_2' = y_1 + 5y_2, & y_2(0) = 7 \end{cases}$$

$$\begin{cases} sY_1 + 3 = 5Y_1 + Y_2 \\ sY_2 - 7 = Y_1 + 5Y_2 \end{cases}$$

$$\begin{cases} (s-5)Y_1 - Y_2 = -3 \\ -Y_1 + (s-5)Y_2 = 7 \end{cases}.$$

$$Y_1 = \frac{-3(s-5)+7}{(s-5)^2 - 1}, \quad Y_2 = \frac{7(s-5)-3}{(s-5)^2-1}$$

$$\begin{aligned}
y_1 &= -3e^{5t}\cosh t + 7e^{5t}\sinh t \\
&= -3e^{5t}\frac{e^t + e^{-t}}{2} + 7e^{5t}\frac{e^t - e^{-t}}{2} \\
&= 2e^{6t} - 5e^{4t} \\
y_2 &= 7e^{5t}\cosh t - 3e^{5t}\sinh t \\
&= 7e^{5t}\frac{e^t + e^{-t}}{2} - 3e^{5t}\frac{e^t - e^{-t}}{2} \\
&= 2e^{6t} + 5e^{4t}
\end{aligned}$$