Math 6171 Solutions to Problems in Test I Fall 2002

1. **(a)** Power series method may not work.
   Frobenius method may not work.

   (b) Power series method may not work.
   Frobenius method works.

   (c) Power series method works.
   Frobenius method works.

2. Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$\left(x - x^2\right) \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} + (2-4x) \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} - 2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0,$$

$$\sum_{n=0}^{\infty} \left[a_n (n+r)(n+r-1) + 2a_n (n+r)\right] x^{n+r-1}$$

$$- \sum_{n=0}^{\infty} \left[a_n (n+r)(n+r-1) + 4a_n (n+r) + 2a_n\right] x^{n+r} = 0,$$

$$\sum_{n=0}^{\infty} \left[a_n (n+r)(n+r+1)\right] x^{n+r-1} - \sum_{n=0}^{\infty} a_n \left[(n+r)(n+r+3) + 2\right] x^{n+r} = 0,$$

$$x^{r-1} : a_0 r (r+1) = 0, \quad \Rightarrow \quad r_1 = 0, \quad r_2 = -1.$$

Let $r_1 = 0$, for x^{n+r-1}, $n = 1, 2, \cdots$, we have

$$\begin{aligned} a_n n (n+1) - a_{n-1} \left[(n-1)(n+2) + 2\right] &= 0, \\ a_n n (n+1) - a_{n-1} \left(n^2 + n\right) &= 0. \end{aligned}$$

$$\begin{aligned} a_n &= a_{n-1}, \quad n = 1, 2, \cdots, \\ y_1 &= a_0 + a_1 x + a_2 x^2 + \cdots = a_0 \frac{1}{1-x}. \end{aligned}$$

3. $y'' + \lambda y = 0, y'(0) = y'(1) = 0.$

Case a

$$\begin{aligned} &\lambda > 0, \qquad y = c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right), \\ &y' = -c_1 \sqrt{\lambda} \sin\left(\sqrt{\lambda} x\right) + c_2 \sqrt{\lambda} \cos\left(\sqrt{\lambda} x\right), \\ &y'(0) = c_2 \sqrt{\lambda} \cos 0 = 0 \Rightarrow c_2 = 0, \\ &y'(1) = c_1 \sqrt{\lambda} \sin \sqrt{\lambda} = 0 \Rightarrow \sqrt{\lambda} = n\pi, \quad n = 1, 2, \cdots, \\ &\text{eigenvalue } \lambda = n^2 \pi^2, \\ &\text{eigenfunction } y = \cos(n\pi x), \quad n = 1, 2, \cdots. \end{aligned}$$

Case b.

$$\begin{array}{l} \lambda = 0, \qquad y = c_1 + c_2 x = 0, \\ y' = c_2, \qquad y'(0) = 0 \Rightarrow c_2 = 0, \\ \qquad\qquad\quad y'(1) = 0 \Rightarrow c_2 = 0. \\ \therefore y = c_1, \\ \text{eigenvalue } \lambda = 0, \\ \text{eigenfunction } y = 1. \end{array}$$

Case c.

$$\begin{array}{l} \lambda < 0, \qquad y = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}, \\ y' = c_1\sqrt{-\lambda}e^{\sqrt{-\lambda}x} - c_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}x}, \\ \begin{cases} y'(0) = c_1\sqrt{-\lambda} - c_2\sqrt{-\lambda} = 0, \\ y'(1) = c_1\sqrt{-\lambda}e^{\sqrt{-\lambda}} - c_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}} = 0, \end{cases} \\ \begin{cases} c_1 - c_2 = 0, \\ c_1 e^{\sqrt{-\lambda}} - c_2 e^{-\sqrt{-\lambda}} = 0, \end{cases} \quad \Rightarrow c_1 = c_2 = 0 \text{ since } e^{\sqrt{-\lambda}} - e^{-\sqrt{-\lambda}} \neq 0. \end{array}$$

Therefore eigenvalues are $0, \pi^2, 4\pi^2, \cdots, n^2\pi^2, \cdots$, and eigenfunctions are 1, $\cos \pi x$, $\cos 2\pi x$, $\cdots$, $\cos n\pi x$, $\cdots$.

4. (a)

$$F(s) = \frac{2}{s^3} + \frac{3}{(s-2)^2 + 3^2} + e^{-4s}.$$

(b)

$$F(s) = \mathcal{L}\left\{\sin^2 t\right\} = \mathcal{L}\left\{\frac{1}{2}(1 - \cos 2t)\right\} = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right).$$

(c)

$$\begin{aligned} F(s) &= \mathcal{L}\{t - tu(t-1)\} = \mathcal{L}\{t - (t-1)u(t-1) - u(t-1)\} \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}. \end{aligned}$$

(d)

$$\begin{aligned} F(s) &= \mathcal{L}\{u(t-3)\cos(t-3+3)\} \\ &= \mathcal{L}\{\cos 3u(t-3)\cos(t-3) - \sin 3u(t-3)\sin(t-3)\} \\ &= \cos 3\frac{se^{-3s}}{s^2+1} - \sin 3\frac{e^{-3s}}{s^2+1}. \end{aligned}$$

5. (a)

$$\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+4}\right\} = \cos 2t + 2\sin 2t.$$

(b)

$$\frac{1}{(s-2)(s+1)} = \frac{A(s+1) + B(s-2)}{(s-2)(s+1)},$$

Let $s=2$, we have

$$1=3A,\quad A=\frac{1}{3}.$$

Let $s=-1$, we have

$$1=-3B,\quad B=\frac{-1}{3}.$$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s+1)}\right\} &= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}-\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}\\ &= \frac{1}{3}e^{2t}-\frac{1}{3}e^{-t}.\end{aligned}$$

(c)

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+10}\right\}=\mathcal{L}\left\{\frac{1}{(s+1)^2+9}\right\}=\frac{1}{3}e^{-t}\sin 3t.$$

(d)

$$\mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+9}\right\}=\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}\Big|_{t\Rightarrow t-2}\cdot u\left(t-2\right)=\cos(3(t-2))u\left(t-2\right).$$

6.

$$s^2Y-1-sY-12Y=\frac{1}{s}e^{-5s},$$

$$Y=\frac{\frac{1}{s}e^{-5s}+1}{s^2-s-12}=\frac{e^{-5s}+s}{s\left(s+3\right)\left(s-4\right)}=\frac{e^{-5s}}{s\left(s+3\right)\left(s-4\right)}+\frac{1}{\left(s+3\right)\left(s-4\right)},$$

$$\begin{aligned}\frac{1}{s\left(s+3\right)\left(s-4\right)} &= \frac{A}{s}+\frac{B}{s+3}+\frac{C}{s-4}\\ &= \frac{A\left(s+3\right)\left(s-4\right)+Bs\left(s-4\right)+Cs\left(s+3\right)}{s\left(s+3\right)\left(s-4\right)}.\end{aligned}$$

$$\begin{aligned}\text{Set}\quad & s=0, && \Rightarrow\ A=\tfrac{-1}{12},\\ & s=-3, && \Rightarrow\ B=\tfrac{1}{21},\\ & s=4, && \Rightarrow\ C=\tfrac{1}{28}.\end{aligned}$$

$$\frac{1}{\left(s+3\right)\left(s-4\right)}=\frac{A\left(s-4\right)+B\left(s+3\right)}{\left(s+3\right)\left(s-4\right)}.$$

$$\begin{aligned}\text{Set}\quad & s=-3, && \Rightarrow\ A=\tfrac{-1}{7},\\ & s=4, && \Rightarrow\ B=\tfrac{1}{7}.\end{aligned}$$

$$\begin{aligned} y(t) &= \left. \left(-\frac{1}{12} + \frac{1}{21} e^{-3t} + \frac{1}{28} e^{4t} \right) \right|_{t \Rightarrow t-5} u(t-5) + \frac{1}{7} e^{4t} - \frac{1}{7} e^{-3t} \\ &= \left(-\frac{1}{12} + \frac{1}{21} e^{-3(t-5)} + \frac{1}{28} e^{4(t-5)} \right) u(t-5) + \frac{1}{7} e^{4t} - \frac{1}{7} e^{-3t}. \end{aligned}$$