Math 6171 Solutions to Problems in Test 2 Fall 2001

1. (a)

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \vec{j}, \qquad \text{Area} = \frac{1}{2}.$$

(b)

$$\begin{vmatrix} 1 & 0 & -7 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} = -1 + (-7)(-2) = 13.$$

(c)

$$\int_0^6 \sqrt{1^2 + \left(2t^{\frac{1}{2}}\right)^2} dt = \int_1^{25} u^{\frac{1}{2}} \cdot \frac{du}{4} = \frac{1}{4} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_1^{25} = \frac{124}{6} = \frac{62}{3}.$$

2. (a)

$$z + 2(y - z) + xy = 2y - z + xy.$$

(b)

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{ay} & \dfrac{\partial}{\partial z} \\ xy^2z^3 & \sin x & 0 \end{vmatrix} = 3xy^2z^2\vec{j} + \left(\cos x - 2xyz^3\right)\vec{k}.$$

(c)

$$\text{grad } f = 6x\vec{i} + 4y\vec{j} + 2z\vec{k}$$

At the point $(1, 2, 3)$

$$\begin{aligned} \text{grad } f &= 6\vec{i} + 8\vec{j} + 6\vec{k} \\ \frac{1}{|\vec{a}|}\vec{a} \cdot \text{grad } f &= \frac{20}{\sqrt{3}}. \end{aligned}$$

3. (a)

$$\operatorname{div}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_1 & u_2 & u_3 \end{vmatrix} = \frac{\partial}{\partial x}\left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)$$
$$= 0.$$

(b)

$$\begin{aligned} \operatorname{curl}(f\vec{v}) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ fv_1 & fv_2 & fv_3 \end{vmatrix} \\ &= \left(f\frac{\partial v_3}{\partial x} + v_3\frac{\partial f}{\partial x} - f\frac{\partial v_2}{\partial z} - v_2\frac{\partial f}{\partial y}\right)\vec{i} \\ &\quad + \left(f\frac{\partial v_1}{\partial z} + v_1\frac{\partial f}{\partial z} - f\frac{\partial v_3}{\partial x} - v_3\frac{\partial f}{\partial x}\right)\vec{j} \\ &\quad + \left(f\frac{\partial v_2}{\partial x} + v_2\frac{\partial f}{\partial x} - f\frac{\partial v_1}{\partial y} - v_1\frac{\partial f}{\partial y}\right)\vec{k} \\ &= f\operatorname{curl}\vec{v} + \operatorname{grad} f \times \vec{v}. \end{aligned}$$

4.

$$\begin{aligned} \iint_S \left(y^2\vec{i} + x^2\vec{j} + z^2\vec{k}\right)\cdot\vec{n}dA &= \iiint_T 2zdxdydz = \int_0^1\int_0^4\int_0^2 2zdzdydx \\ &= 4\cdot 4 = 16. \end{aligned}$$

5.

$$\begin{aligned} \operatorname{curl}\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2 + ye^{xy} & xe^{xy} - z\sin(yz) & -y\sin(yz) - e^{2z} \end{vmatrix} \\ &= [-\sin(yz) - yzcos(yz) + \sin(yz) + zy\cos(yz)]\,\vec{i} \\ &\quad + \left(e^{xy} + xye^{xy} - e^{xy} - yxe^{xy}\right)\vec{k} \\ &= 0. \end{aligned}$$

Hence, it has a potential function

$$\frac{\partial f}{\partial x} = 3x^2 + ye^{xy}$$

$$
\begin{aligned}
f &= x^3 + e^{xy} + g(y,z) \\
\frac{\partial f}{\partial y} &= xe^{xy} - z\sin(yz) = xe^{xy} + \frac{\partial g}{\partial y} \\
\frac{\partial g}{\partial y} &= -z\sin(yz) \\
g &= \cos(yz) + h(z) \\
f &= x^3 + xe^{xy} + \cos(yz) + h(z) \\
\frac{\partial f}{\partial z} &= -y\sin(yz) + \frac{dh}{dz} = -y\sin(yz) - e^{2z} \\
\frac{dh}{dz} &= -e^{2z} \\
h &= \frac{-1}{2}e^{2z} \\
f &= x^3 + e^{xy} + \cos(yz) - \frac{-1}{2}e^{2z} \\
\int\limits_{(0,0,1)}^{(1,2,3)} \vec{F}\cdot d\vec{r} &= 1 + e^2 + \cos 6 - \frac{1}{2}e^6 - \left[1 + 1 - \frac{1}{2}e^2\right] \\
&= \frac{3}{2}e^2 + \cos 6 - \frac{1}{2}e^6 - 1.
\end{aligned}
$$

6.

$$
\begin{aligned}
\vec{r} &= u\cos v\vec{i} + u\sin v\vec{j} + u^2\vec{k}, \qquad 0 \le u \le 2, \quad 0 \le v \le 2\pi \\
\vec{r}_u &= \cos v\vec{i} + \sin v\vec{j} + 2u\vec{k} \\
\vec{r}_v &= -u\sin v\vec{i} + u\cos v\vec{j}.
\end{aligned}
$$

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\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = -2u^2\cos v\vec{i} - 2u^2\sin v\vec{j} + u\vec{k}
$$

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\vec{F}\cdot(\vec{r}_u \times \vec{r}_v) = -2u^3\cos^2 v - 2u^3\sin^2 v + 3u^5
$$

$$
\begin{aligned}
\iint\limits_S \vec{F}\cdot\vec{n}dA &= \int\limits_0^{2\pi}\int\limits_0^{2} \left(-2u^3 + 3u^5\right) du dv = 2\pi\left(-\frac{u^4}{2} + \frac{u^6}{2}\right)\Bigg|_0^2 \\
&= 2\pi\cdot 24 = 48\pi.
\end{aligned}
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7.

$$
S : x + y + z = 2
$$

$$
\begin{aligned}
\vec{r} &= x\vec{i} + y\vec{j} + (2-x-y)\,\vec{k}, \quad 0 \le x \le 2, \quad 0 \le y \le 2-x \\
\vec{r}_x &= \vec{i} - \vec{k} \\
\vec{r}_y &= \vec{j} - \vec{k} \\
\vec{r}_x \times \vec{r}_y &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i} + \vec{j} + \vec{k} \\
\text{curl}\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y+2 & 3x+4 & 5z \end{vmatrix} = 2\vec{k}
\end{aligned}
$$

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\begin{aligned}
I &= \oint\limits_C \vec{F} \cdot d\vec{r} = \iint\limits_S \text{curl } \vec{F} \cdot \vec{n} dA = \int\limits_0^2 \int\limits_0^{2-x} \text{curl } \vec{F} \cdot (\vec{r}_x \times \vec{r}_y) dy dx \\
&= \int\limits_0^2 \int\limits_0^{2-x} 2 dy dx = 4.
\end{aligned}
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