Math 6171 Solutions to Problems in Test 2 Fall 2002

1. (a)

$$\begin{aligned}\operatorname{grad}\left(\operatorname{div}\left(x^2z\mathbf{i}+(y-z)^2\mathbf{j}+xy\mathbf{k}\right)\right) &= \operatorname{grad}\left(2xz+2(y-z)\right)\\ &= 2z\mathbf{i}+2\mathbf{j}+(2x-2)\mathbf{k}\end{aligned}$$

(b)

$$\begin{aligned}\operatorname{curl}\left(\cos(yz)\,\mathbf{i}+\sin x\mathbf{j}\right) &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \cos(yz) & \sin x & 0\end{vmatrix}\\ &= -y\sin(yz)\,\mathbf{j}+(\cos x+z\sin(yz))\,\mathbf{k}\end{aligned}$$

(c)

$$\begin{aligned}f(x,y,z) &= z-\sqrt{x^2+y^2}\\ \operatorname{grad} f &= -\frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}}\cdot 2x\mathbf{i}-\frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}}\cdot 2y\mathbf{j}+\mathbf{k}\\ \mathbf{N}(p) &= -\frac{3}{5}\mathbf{i}-\frac{4}{5}\mathbf{j}+1\mathbf{k}\\ \mathbf{n} &= \frac{1}{\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2+1}}\left(-\frac{3}{5}\mathbf{i}-\frac{4}{5}\mathbf{j}+\mathbf{k}\right)\\ &= -\frac{3}{5\sqrt{2}}\mathbf{i}-\frac{4}{5\sqrt{2}}\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k}.\end{aligned}$$

2. (a)

$$\begin{vmatrix}1 & 2 & 3\\ 1 & 2 & 4\\ 2 & 2 & 5\end{vmatrix}=\begin{vmatrix}1 & 2 & 3\\ 0 & 0 & 1\\ 2 & 2 & 5\end{vmatrix}=-\begin{vmatrix}1 & 2\\ 2 & 2\end{vmatrix}=2.$$

(b)

$$\mathbf{N}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 2\\ 2 & 1 & -1\end{vmatrix}=-3\mathbf{i}+5\mathbf{j}-\mathbf{k}$$

$$-3x+5(y-1)-(z-1)=0$$

$$-3x+5y-z=4.$$

(c)

$$\frac{x^2}{\frac{9}{4}} + \frac{(y-1)^2}{9} = 1, \quad \left(\frac{x}{\frac{3}{2}}\right)^2 + \left(\frac{y-1}{3}\right)^2 = 1$$
$$x = \frac{3}{2}\cos\theta, \quad y = 1 + 3\sin\theta, \quad z = 1 + 3\sin\theta.$$

(d)

$$\begin{aligned} L &= \int_0^4 \sqrt{\mathbf{r}' \cdot \mathbf{r}'} dt = \int_0^4 \sqrt{1^2 + \left(\frac{3}{2}\right)^2 t}\, dt \\ &= \int_1^{10} \eta^{\frac{1}{2}} \cdot \frac{4}{9} d\eta = \frac{4}{9}\frac{\eta^{\frac{3}{2}}}{\frac{3}{2}}\bigg|_1^{10} = \frac{8}{27}\left(10^{3/2} - 1\right) \\ &= 9.073415 \end{aligned}$$

3. (a)

$$\begin{aligned} \operatorname{curl}(\operatorname{grad} f) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\ &= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k} \\ &= 0 \end{aligned}$$

(b)

$$\begin{aligned} \operatorname{curl}(\mathbf{u} + \mathbf{v}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_1 + v_1 & u_2 + v_2 & u_3 + v_3 \end{vmatrix} \\ &= \left[\frac{\partial}{\partial y}(u_3 + v_3) - \frac{\partial}{\partial z}(u_2 + v_2)\right]\mathbf{i} \\ &\quad + \left[\frac{\partial}{\partial z}(u_1 + v_1) - \frac{\partial}{\partial x}(u_2 + v_2)\right]\mathbf{j} \\ &\quad + \left[\frac{\partial}{\partial x}(u_2 + v_2) - \frac{\partial}{\partial y}(u_1 + v_1)\right]\mathbf{k} \\ &= \left(\frac{\partial}{\partial y}u_3 - \frac{\partial}{\partial z}u_2\right)\mathbf{i} + \left(\frac{\partial}{\partial z}u_1 - \frac{\partial}{\partial x}u_2\right)\mathbf{j} + \left(\frac{\partial}{\partial x}u_2 - \frac{\partial}{\partial y}u_1\right)\mathbf{k} \\ &\quad + \left(\frac{\partial}{\partial y}v_3 - \frac{\partial}{\partial z}v_2\right)\mathbf{i} + \left(\frac{\partial}{\partial z}v_1 - \frac{\partial}{\partial x}v_2\right)\mathbf{j} + \left(\frac{\partial}{\partial x}v_2 - \frac{\partial}{\partial y}v_1\right)\mathbf{k} \\ &= \operatorname{curl}\mathbf{u} + \operatorname{curl}\mathbf{v}. \end{aligned}$$

4.

$$\begin{aligned}\mathbf{F} &= [2x, y, -z] = [4t, \cos t, -\sin t] \\ d\mathbf{r} &= [2, -\sin t, \cos t]\, dt\end{aligned}$$

$$\begin{aligned}\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_0^{2\pi} (8t - 2\cos t \sin t)\, dt = 4t^2 + \left.\frac{\cos 2t}{2}\right|_0^{2\pi} \\ &= 16\pi^2.\end{aligned}$$

5.

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \quad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi, \\ z = r^2. \end{cases}$$

$$\begin{aligned}\mathbf{r}_r &= \cos\theta\mathbf{i} + \sin\theta\mathbf{j} + 2r\mathbf{k} \\ \mathbf{r}_\theta &= -r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j}\end{aligned}$$

$$\mathbf{N} = \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = -2r^2\cos\theta\mathbf{i} - 2r^2\sin\theta\mathbf{j} + r\mathbf{k}$$

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j} + 3z^2\mathbf{k} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + 3r^4\mathbf{k}$$

$$\begin{aligned}\iint_S \mathbf{F} \cdot \mathbf{n} dA &= \int_0^{2\pi}\int_0^{2} \left(-2r^3\cos^2\theta - 2r^3\sin^2\theta + 3r^5\right) dr d\theta \\ &= 2\pi \int_0^2 \left(3r^5 - 2r^3\right) dr = 2\pi \left.\left(\frac{1}{2}r^6 - \frac{1}{2}r^4\right)\right|_0^2 \\ &= \pi(64 - 16) = 48\pi.\end{aligned}$$

6.

$$\begin{aligned}\text{curl}\,\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x + ye^{xy} & xe^{xy} + z\cos(yz) & y\cos(yz) + z^2 \end{vmatrix} \\ &= \{\cos(yz) - yz\sin(yz) - [\cos(yz) - zy\sin(yz)]\}\,\mathbf{i} \\ &\quad + 0\mathbf{j} + [e^{xy} + xye^{xy} - (e^{xy} + xye^{xy})]\,\mathbf{k} \\ &= 0\end{aligned}$$

$$
\begin{aligned}
f_x &= x + ye^{xy} \\
f(x,y,z) &= \int (x + ye^{xy})\,dx = \frac{x^2}{2} + e^{xy} + g(y,z) \\
\frac{\partial f}{\partial y} &= xe^{xy} + \frac{\partial g}{\partial y} = xe^{xy} + z\cos(yz) \\
g(y,z) &= \int z\cos(yz)\,dy = \sin(yz) + h(z) \\
f(x,y,z) &= \frac{x^2}{2} + e^{xy} + \sin(yz) + h(z) \\
\frac{\partial f}{\partial z} &= y\cos(yz) + \frac{dh}{dz} = y\cos(yz) + z^2 \\
h &= \frac{z^3}{3} \\
f(x,y,z) &= \frac{x^2}{2} + e^{xy} + \sin(yz) + \frac{z^3}{3} \\
\int_{(0,0,0)}^{(1,1,1)} \mathbf{F}\cdot d\mathbf{r} &= \frac{1}{2} + e + \sin 1 + \frac{1}{3} - 1 \\
&= e + \sin 1 - \frac{1}{6} = 3.3931.
\end{aligned}
$$

7.

$$
\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}dA \\
\text{curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2z + e^{x^3} & -x & x \end{vmatrix} = (2-1)\,\mathbf{j} - \mathbf{k} = \mathbf{j} - \mathbf{k}
\end{aligned}
$$

$$
\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r\sin\theta + 1 \end{cases} \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.
$$

$$
\begin{aligned}
\mathbf{r}_r &= \cos\theta\mathbf{i} + \sin\theta\mathbf{j} + \sin\theta\mathbf{k} \\
\mathbf{r}_\theta &= -r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j} + r\cos\theta\mathbf{k}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{r}_r \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & \sin\theta \\ -r\sin\theta & r\cos\theta & r\cos\theta \end{vmatrix} \\
&= \left(-r\sin^2\theta - r\cos^2\theta\right)\mathbf{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\mathbf{k} \\
&= -r\mathbf{j} + r\mathbf{k}
\end{aligned}
$$

$$
\iint\limits_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} dA = \int\limits_0^{2\pi} \int\limits_0^1 -2r dr d\theta = -2\pi \cdot r^2\Big|_0^1 = -2\pi.
$$