Math 6171 Solutions to Problems in Test III Fall 2001

1. (a) odd;
   (b) even;
   (c) neither odd nor even;
   (d) even.

2. (a)

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^\pi |x|\,dx = \frac{1}{\pi}\frac{x^2}{2}\bigg|_0^\pi = \frac{\pi}{2},\\
a_n &= \frac{2}{\pi}\int_0^\pi x\cos nx dx = \frac{2}{\pi}\left[x\cdot\frac{\sin nx}{n}\bigg|_0^\pi - \int_0^\pi \frac{\sin nx}{n}dx\right]\\
&= \frac{2}{\pi}\frac{\cos nx}{n^2}\bigg|_0^\pi = \frac{2}{n^2\pi}\left[(-1)^n-1\right] = \begin{cases} \frac{-4}{n^2\pi}, & n=1,3,\cdots,\\ 0, & n=2,4,\cdots,\end{cases}\\
|x| &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos(2m+1)x}{(2m+1)^2}.
\end{aligned}
$$

   (b) Let $x=0$

$$
0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2},
$$

$$
\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.
$$

3. Since u depends r only, we have

$$
\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} = 0.
$$

Let $w = \dfrac{du}{dr}$, the equation can be written as

$$
\begin{aligned}
\frac{dw}{dr} + \frac{2}{r}w &= 0 \quad\Rightarrow\quad \frac{dw}{w} = -\frac{2dr}{r} \quad\Rightarrow\quad w = cr^{-2},\\
\frac{du}{dr} &= \frac{c}{r^2} \quad\Rightarrow\quad u = \int\frac{cdr}{r^2} = -\frac{c}{r}+k.
\end{aligned}
$$

4. (a)

$$\mathcal{L}\left\{\frac{\partial u}{\partial x}+x^2\frac{\partial u}{\partial t}\right\}=\frac{\partial U}{\partial x}+x^2\left(sU-u(x,0)\right)=0,$$

$$\frac{\partial U}{\partial x}+sx^2U=0,\qquad \frac{dU}{U}=-sx^2dx,\qquad \ln U=\frac{-sx^3}{3}+c,$$

$$U\left(x,s\right)=e^ce^{-\frac{x^3}{3}s}\Rightarrow U\left(x,s\right)=U\left(0,s\right)e^{-\frac{x^3}{3}s},$$

$$\begin{aligned}u\left(x,t\right) &= \mathcal{L}^{-1}\left\{U\left(0,s\right)e^{-\frac{x^3}{3}s}\right\}=u\left(t-\frac{x^3}{3}\right)\mathcal{L}^{-1}\left\{U\left(0,s\right)\right\}|_{t\to t-\frac{x^3}{3}}\\ &= u\left(t-\frac{x^3}{3}\right)\left(t-\frac{x^3}{3}\right)^3.\end{aligned}$$

(b)

$$u\left(x,t\right)=u\left(t-\frac{x^3}{3}\right)e^{\cos\left(t-\frac{x^3}{3}\right)}.$$

5. (a)

$$\begin{aligned}y' &= 0 \text{ or } xy'+2y=0.\\ y' &= 0 \quad\Rightarrow\quad y=\text{constant}.\\ xy'+2y &= 0 \quad\Rightarrow\quad y'=-\frac{2y}{x},\quad \frac{dy}{y}=-\frac{2dx}{x},\qquad \ln y=\ln x^{-2}+c,\\ &\qquad\qquad x^2y= \text{ constant}.\end{aligned}$$

(b) Let $v=y,\quad z=x^2y$. Since

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial U}{\partial z}\frac{\partial z}{\partial x}=2xy\frac{\partial U}{\partial z},\\ \frac{\partial^2 u}{\partial x^2} &= 2y\frac{\partial U}{\partial z}+2xy\frac{\partial^2 U}{\partial z^2}\cdot 2xy,\\ \frac{\partial^2 u}{\partial x\partial y} &= 2x\frac{\partial U}{\partial z}+2xy\left[\frac{\partial^2 U}{\partial z\partial v}+\frac{\partial^2 U}{\partial z^2}\cdot x^2\right],\end{aligned}$$

we have

$$\begin{aligned}& x\left(2y\frac{\partial U}{\partial z}+4x^2y^2\frac{\partial^2 U}{\partial z^2}\right)-2y\left[2x\frac{\partial U}{\partial z}+2xy\frac{\partial^2 U}{\partial z\partial v}+2x^3y\frac{\partial^2 U}{\partial z^2}\right]+2xy\frac{\partial U}{\partial z}\\ =\ & -4xy^2\frac{\partial^2 U}{\partial z\partial v}=0 \quad\Rightarrow\quad \frac{\partial^2 U}{\partial z\partial v}=0.\end{aligned}$$

(c)

$$\frac{\partial U}{\partial z} = f(z), \qquad U = \int f(z)\,dz + g(v) = F(z) + g(v),$$
$$u(x,y) = F\left(x^2 y\right) + g(y).$$

6. Let $u = F(x)\,G(t)$, we have

$$\begin{aligned} F(x)\frac{dG}{dt} &= c^2 G(t)\frac{d^2F}{dx^2}, \\ \frac{1}{c^2 G}\frac{dG}{dt} &= \frac{1}{F}\frac{d^2F}{dx^2} = -k, \qquad F'(0)\,G(t) = F'(L)\,G(t) = 0 \Rightarrow F'(0) = F'(L) = 0, \end{aligned}$$

$$\begin{cases} F'' + kF = 0, \\ F'(0) = F'(L) = 0. \end{cases}$$

(a)

$$\begin{aligned} k = 0: \ & F(x) = ax + b, \quad F'(x) = a, \quad F'(0) = F'(L) = a = 0, \\ & F_0(x) = b, \quad G(t) = 1 \quad \Rightarrow \quad u_0 = c_0. \end{aligned}$$

(b)

$$\begin{aligned} k < 0: \ & F(x) = c_1 e^{\sqrt{-k}x} + c_2 e^{-\sqrt{-k}x}, \quad F'(x) = c_1\sqrt{-k}e^{\sqrt{-k}x} - c_2\sqrt{-k}e^{-\sqrt{-k}x}, \\ & F(0) = c_1\sqrt{-k} - c_2\sqrt{-k} = 0, \\ & F(L) = c_1\sqrt{-k}e^{\sqrt{-k}L} - c_2\sqrt{-k}e^{-\sqrt{-k}L} = 0, \quad \Rightarrow c_1 = c_2 = 0, \\ & \text{no solution.} \end{aligned}$$

(c)

$$\begin{aligned} k > 0: \ & F(x) = c_1 \cos\sqrt{k}x + c_2 \sin\sqrt{k}x, \\ & F'(x) = -c_1\sqrt{k}\sin\sqrt{k}x + c_2\sqrt{k}\cos\sqrt{k}x, \\ & F'(0) = -c_2\sqrt{k} = 0 \Rightarrow c_2 = 0, \\ & F'(L) = -c_1\sqrt{k}\sin\sqrt{k}L = 0, \quad \Rightarrow \quad \sqrt{k}L = n\pi, \quad n = 1, 2, \cdots, \\ & F_n = c_n \cos\frac{n\pi}{L}x, \\ & \frac{dG}{G} = -c^2\left(n\pi/L\right)^2, \qquad \ln G = -\left(cn\pi/L\right)^2 t, \qquad G = e^{-(cn\pi/L)t}, \\ & u_n = c_n \cos\frac{n\pi}{L}x e^{-(cn\pi/L)^2 t} \end{aligned}$$

Therefore

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi}{L}x e^{-(cn\pi/L)^2 t}.$$

Since

$$f(x) = u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi}{L}x,$$

c_0 and c_n are given by

$$
\begin{aligned}
c_0 &= \frac{1}{L}\int_0^L f(x)\,dx, \\
c_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx, \qquad n = 1, 2, \cdots .
\end{aligned}
$$