Math 6171 Solutions to Problems in Test III Fall 2002

1. (a) even;
   (b) odd;
   (c) neither odd nor even;
   (d) π;
   (e) 4π.

2. (a)

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^\pi \sin x dx = \frac{-1}{\pi}\cos x\Big|_0^\pi = \frac{-1}{\pi}(-1-1) = \frac{2}{\pi}, \\
a_n &= \frac{2}{\pi}\int_0^\pi \sin x \cos nx dx = \frac{1}{\pi}\int_0^\pi [\sin(1+n)x + \sin(1-n)x]\, dx \\
&= \begin{cases} \dfrac{1}{\pi}\left[\dfrac{-\cos(1+n)x}{1+n} + \dfrac{-\cos(1-n)x}{1-n}\right]\Bigg|_0^\pi, & n \neq 1 \\ \dfrac{1}{\pi}\cdot\dfrac{-\cos 2x}{2}\Big|_0^\pi, & n = 1 \end{cases} \\
&= \begin{cases} 0, & n\text{—odd} \\ \dfrac{1}{\pi}\left[\dfrac{2}{1+n} + \dfrac{2}{1-n}\right], & n\text{—even} \end{cases} \\
&= \begin{cases} 0, & n\text{—odd} \\ \dfrac{1}{\pi}\cdot\dfrac{-4}{(n-1)(n+1)}, & n\text{—even,} \end{cases} \\
\sin x &= \frac{2}{\pi}\left[1 - \frac{2}{1\cdot 3}\cos 2x - \frac{2}{3\cdot 5}\cos 4x - \frac{2}{5\cdot 7}\cos 6x - \cdots\right].
\end{aligned}$$

   (b) Let $x = 0$. We have

$$0 = \frac{2}{\pi}\left[1 - \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} - \frac{2}{5\cdot 7} - \cdots\right] = 0$$

   or

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \frac{1}{2}.$$

3. Let $\cos\phi = x$.

$$\begin{aligned}
f(\phi) &= 1 - \cos\phi - 6\cos^2\phi \\
&= 1 - x - 6x^2 \\
&= -4P_2 - P_1 - P_0 \equiv F(x).
\end{aligned}$$

$$\begin{aligned}
u &= \sum_{n=0}^{\infty} A_n r^n P_n(\cos\phi) \\
A_0 &= \frac{1}{2}\int_{-1}^{1} F(x)\, P_0(x)\, dx = \frac{-1}{2}\int_{-1}^{1} P_0^2(x)\, dx = -1 \\
A_1 &= \frac{3}{2}\int_{-1}^{1} F(x)\, P_1(x)\, dx = \frac{-3}{2}\int_{-1}^{1} P_1^2(x)\, dx = -1 \\
A_2 &= \frac{5}{2}\int_{-1}^{1} F(x)\, P_2(x)\, dx = -10\int_{-1}^{1} P_2^2(x)\, dx = -4 \\
A_n &= \frac{2n+1}{2}\int_{-1}^{1} F(x)\, P_n(x)\, dx = 0, \quad n = 3, 4, \cdots,
\end{aligned}$$

$$u = -1 - r\cos\phi - 4r^2\left[\frac{1}{2}\left(3\cos^2\phi - 1\right)\right]$$

4. (a)

$$(y')^2 + 5y' + 4 = (y'+1)(y'+4) = 0.$$

$$\begin{aligned}
y' = -1, \quad & y = -x + c, \quad y + x = c, \\
y' = -4, \quad & y = -4x + c, \quad y + 4x = c.
\end{aligned}$$

(b)

$$v = x + y, \quad z = 4x + y$$

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial U}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial U}{\partial v} + 4\frac{\partial U}{\partial z}, \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial v} + 4\frac{\partial U}{\partial z}\right) \\
&= \frac{\partial}{\partial v}\left(\frac{\partial U}{\partial v} + 4\frac{\partial U}{\partial z}\right) + 4\frac{\partial}{\partial z}\left(\frac{\partial U}{\partial v} + 4\frac{\partial U}{\partial z}\right) \\
&= \frac{\partial^2 U}{\partial v^2} + 8\frac{\partial^2 U}{\partial z \partial v} + 16\frac{\partial^2 U}{\partial z^2}, \\
\frac{\partial u}{\partial y} &= \frac{\partial U}{\partial v} + \frac{\partial U}{\partial z}, \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 U}{\partial v^2} + 2\frac{\partial^2 U}{\partial v \partial z} + \frac{\partial^2 U}{\partial z^2}, \\
\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial v}\left(\frac{\partial U}{\partial v} + \frac{\partial U}{\partial z}\right) + 4\frac{\partial}{\partial z}\left(\frac{\partial U}{\partial v} + \frac{\partial U}{\partial z}\right) \\
&= \frac{\partial^2 U}{\partial v^2} + 5\frac{\partial^2 U}{\partial v \partial z} + 4\frac{\partial^2 U}{\partial z^2}.
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} \\
= &\ (1-5+4)\frac{\partial^2 U}{\partial v^2} + (8-25+8)\frac{\partial^2 U}{\partial v \partial z} + (16-20+4)\frac{\partial^2 U}{\partial z^2} \\
= &\ -9\frac{\partial^2 U}{\partial v \partial z} = 0.
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{\partial^2 U}{\partial v \partial z} &= 0, \quad \frac{\partial U}{\partial z} = \int \frac{\partial^2 U}{\partial v \partial z} dv = f(z), \\
U &= \int \frac{\partial U}{\partial z} dz = \int f(z)\, dz = F(z) + G(v), \\
u &= F(4x+y) + G(x+y).
\end{aligned}$$

5. (a)

$$\begin{aligned}
\frac{\partial v}{\partial x} + x^3 \frac{\partial v}{\partial t} &= 6x^3 \\
\mathcal{L}\left\{\frac{\partial v}{\partial x}\right\} + \mathcal{L}\left\{x^3 \frac{\partial v}{\partial t}\right\} &= \mathcal{L}\left\{6x^3\right\} \\
\frac{\partial V(x,s)}{\partial x} + x^3 s V(x,s) &= \frac{6x^3}{s} \\
\mu &= e^{\int x^3 s dx} = e^{x^4 s/4} \\
\frac{\partial e^{\frac{x^4 s}{4}} V(x,s)}{\partial x} &= e^{\frac{x^4 s}{4}} \cdot \frac{6x^3}{s} \\
e^{\frac{x^4 s}{4}} V(x,s)\Big|_0^x &= \int_0^x e^{x^4 s/4} \frac{6x^3}{s} dx \\
e^{\frac{x^4 s}{4}} V(x,s) - V(0,s) &= \frac{6}{s^2} e^{x^4 s/4}\Big|_0^x = \frac{6}{s^2} e^{x^4 s/4} - \frac{6}{s^2} \\
V(x,s) &= V(0,s)\, e^{-x^4 s/4} + \frac{6}{s^2}\left(1 - e^{-x^4 s/4}\right) \\
V(0,s) &= \mathcal{L}\{v(0,t)\} = \mathcal{L}\{f(t)\} \\
V(x,s) &= \mathcal{L}\{f(t)\}\, e^{-x^4 s/4} + \frac{6}{s^2}\left(1 - e^{-x^4 s/4}\right) \\
v(x,t) &= \mathcal{L}^{-1}\{V(x,s)\} \\
&= f\left(t - \frac{x^4}{4}\right) u\left(t - \frac{x^4}{4}\right) + 6t - 6\left(t - \frac{x^4}{4}\right) u\left(t - \frac{x^4}{4}\right).
\end{aligned}$$

If

$$f(t) = t, \text{ then } v(x,t) = 6t - 5\left(t - \frac{x^4}{4}\right) u\left(t - \frac{x^4}{4}\right).$$

(b) If

$$f(t) = e^{\sin 2t},$$

then

$$V(x,t) = e^{\sin 2\left(t - x^4/4\right)} u\left(t - \frac{x^4}{4}\right) + 6t - 6\left(t - \frac{x^4}{4}\right) u\left(t - \frac{x^4}{4}\right).$$

6. Let

$$\begin{aligned} u(x,t) &= F(x)\,G(t) \\ \frac{1}{c^2 G}\frac{dG}{dt} &= \frac{1}{F}\frac{d^2F}{dx^2} = -k, \qquad u(0,t) = u_x(L,t) = 0 \Rightarrow F(0) = \frac{dF}{dx}(L) = 0 \\ \frac{d^2F}{dx^2} + kF &= 0. \end{aligned}$$

Case 1. $k > 0$:

$$\begin{aligned} F(x) &= c_1 \cos\sqrt{k}x + c_2 \sin\sqrt{k}x, \quad F(0) = 0 \Rightarrow c_1 = 0 \\ \frac{dF}{dx}(L) &= c_2\sqrt{k}\cos\sqrt{k}L = 0 \Rightarrow \sqrt{k}L = \left(n + \frac{1}{2}\right)\pi, \quad n = 0, 1, \cdots, \\ F_n &= c_2 \sin\frac{\left(n+\frac{1}{2}\right)\pi}{L}x, \quad n = 0, 1, 2, \cdots, \\ G_n &= e^{-kc^2 t} = e^{-((n+1/2)\pi c/L)^2 t}. \end{aligned}$$

Case 2. $k = 0$:

$$F = c_1 + c_2 x \Rightarrow c_1 = c_2 = 0.$$

Case 3. $k < 0$:

$$F = c_1 e^{\sqrt{-k}x} + c_2 e^{-\sqrt{-k}x} \Rightarrow c_1 = c_2 = 0.$$

Therefore

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin\frac{\left(n+\frac{1}{2}\right)\pi}{L} x e^{-((n+1/2)\pi c/L)^2 t}.$$

Since

$$f(x) = u(x,0) = \sum_{n=0}^{\infty} A_n \sin\frac{\left(n+\frac{1}{2}\right)\pi}{L}x,$$

we have

$$A_n = \frac{2}{L}\int_0^L \sin\frac{\left(n+\frac{1}{2}\right)\pi}{L} x f(x)\,dx.$$

Here we have used the following relations:

$$\int_0^L \sin^2 \frac{\left(n+\frac{1}{2}\right)\pi}{L} x dx = \frac{1}{2}\int_0^L \left(1-\cos\frac{2\left(n+\frac{1}{2}\right)\pi}{L}x\right) dx = \frac{L}{2},$$

$$\int_0^L \sin\frac{\left(m+\frac{1}{2}\right)\pi}{L}x \sin\frac{\left(n+\frac{1}{2}\right)\pi}{L} x dx = \frac{1}{2}\int_0^L \left[\cos\left(\frac{(m-n)\pi}{L}x\right)\right.$$

$$\left. -\cos\left(\frac{(m+n+1)\pi}{L}x\right)\right] dx = 0,\ \text{if } m \neq n.$$