MATH 6172 **Test I** **Spring 2003**

Name : ____________________

SHOW THE DETAILS OF YOUR WORK!! ID : ________________

1. (a) Find Re $\dfrac{z^2}{\bar{z}}$

   (b) Find Im $\dfrac{1}{\bar{z}}$

   (c) Find the polar form of $\dfrac{i}{1+i}$

   (d) Find all roots of $\sqrt[4]{-16}$

   (e) Show that $|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$.

2. Are the following functions harmonic? If so, find a corresponding conjugate function $u(x, y)$ for each of them so that $f(z) = u(x, y) + iv(x, y)$ is an analytic function.

   (a) $v = x^3 - 3xy^2$

   (b) $v = e^x \cos 3y$

3. (a) Find and sketch the image of the region: $\pi/4 < \text{Arg z} < \pi/2$ under the mapping $w = z^2$.

(b) Find and sketch the image of the region: $0 < y < 2$ under the mapping $w = e^z$.

4. (a) Find the linear fractional transformation $w = \dfrac{az+b}{cz+d}$ that maps $-1, 0, 1$ onto $-1, i, 1$ respectively and determine what the lower half-plane is mapped onto.

(b) Find a linear fractional transformation that maps $|z| \leq 1$ onto $|w| \leq 1$ such that $z = i/3$ is mapped onto $w = 0$. (You need to show that the transformation maps $|z| \leq 1$ onto $|w| \leq 1$.)

(c) Show that substituting any linear fractional transformation into a linear fractional transformation gives another linear fractional transformation.

5. Integrate

(a) $\int_C e^{3z} dz$, C is the path from 2 along the axes to i.

(b) $\int_C \text{Im zdz}$, C is an ellipse: $x^2 + y^2/4 = 1$, counterclockwise.

(c) Show that $\bar{z} = x - iy$ is not analytic and calculate $\int_C \bar{z} dz$, C is the unit circle, counterclockwise.

6. Integrate the following $f(z)$ around the contour C in the counterclockwise sense.

(a) $f(z) = \dfrac{4 - \sin z}{z^2 - 2z} + \dfrac{z+1}{z+3}, \quad C : |z| = 1$

(b) $f(z) = \dfrac{1 + \sin z}{(z+1)^2}, \quad C : |z - i| = 2$

(c) $f(z) = \dfrac{e^z}{4z^2 - 1}, \quad C : |z| = 1$

(d) Show that $\oint_C (z - z_1)^{-1}(z - z_2)^{-1} dz = 0$ for a simple closed path C enclosing z_1 and z_2, which are arbitrary.