Math 6172 Answers to Problems in Test I Spring 2003

1. (a)

$$\mathrm{Re}\frac{z^2}{\bar{z}} = \frac{x^3 - 3xy^2}{x^2 + y^2}.$$

(b)

$$\mathrm{Im}\frac{1}{\bar{Z}} = \frac{y}{x^2 + y^2}.$$

(c)

$$\frac{i}{1+i} = \frac{\sqrt{2}}{2}e^{i\pi/4}.$$

(d)

$$\sqrt[4]{-16} = 2e^{i(1+2n)\pi/4}, \qquad n = 0, 1, 2, 3.$$

(e)

$$\begin{aligned} |z_1 + z_2|^2 + |z_1 - z_2|^2 &= (x_1 + x_2)^2 + (y_1 + y_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 \\ &= 2\left[x_1^2 + x_2^2 + y_1^2 + y_2^2\right] = 2\left[|z_1|^2 + |z_2|^2\right]. \end{aligned}$$

2. (a) Yes.

$$\begin{aligned} &u = -3x^2y + y^3 + c, \\ &f(z) = -3x^2y + y^3 + c + i\left(x^3 - 3xy^2\right) = iz^3 + c. \end{aligned}$$

(b) Not.

3. (a)

$$\begin{cases} 0 \le r < \infty, \\ \dfrac{\pi}{4} < \theta < \dfrac{\pi}{2} \end{cases} \Rightarrow \begin{cases} 0 \le R < \infty, \\ \dfrac{\pi}{2} < \psi < \pi. \end{cases}$$

(b)

$$\begin{cases} 0 < y < 2, \\ -\infty < x < \infty \end{cases} \Rightarrow \begin{cases} 0 < R < \infty, \\ 0 < \psi < 2. \end{cases}$$

4. (a)

$$w = \frac{z + i}{iz + 1}.$$

(b)

$$w = \frac{z - \frac{i}{3}}{-\frac{i}{3}z - 1}.$$

(c)

$$\begin{aligned} w_1 &= \frac{a_1 z + b_1}{c_1 z + d_1}, \\ w &= \frac{a w_1 + b}{c w_1 + d} = \frac{a \frac{a_1 z + b_1}{c_1 z + d_1} + b}{c \frac{a_1 z + b_1}{c_1 z + d_1} + d} \\ &= \frac{a\left(a_1 z + b_1\right) + b\left(c_1 z + d_1\right)}{c\left(a_1 z + b_1\right) + d\left(c_1 z + d_1\right)} = \frac{\left(a a_1 + b c_1\right) z + a b_1 + b d_1}{\left(c a_1 + d c_1\right) + c b_1 + d d_1}. \end{aligned}$$

5. (a)

$$\int_c e^{3z} dz = \frac{1}{3}\left(e^{3i} - e^{6}\right).$$

(b)

$$\int_c \mathrm{Im} z dz = -2\pi.$$

(c)

$$\int_c \bar{z} dz = 2\pi i.$$

6. (a)

$$\oint_c \left(\frac{4 - \sin z}{z^2 - 2z} + \frac{1 + z}{z + 3}\right) dz = -4\pi i.$$

(b)

$$\oint_c \frac{(1 + \sin z)}{(z + 1)^2} dz = \cos 1 \cdot 2\pi i.$$

(c)

$$\oint_c \frac{e^z}{4z^2 - 1} dz = \frac{\pi i}{2}\left(e^{\frac{1}{2}} - e^{-\frac{1}{2}}\right).$$

(d)

$$\begin{aligned} \oint_c \frac{1}{\left(z - z_1\right)\left(z - z_2\right)} dz &= \oint_{c_1} \frac{\frac{1}{z - z_2}}{z - z_1} dz + \oint_{c_2} \frac{\frac{1}{z - z_1}}{z - z_2} dz \\ &= 2\pi i \frac{1}{z_1 - z_2} + 2\pi i \frac{1}{z_2 - z_1} = 0. \end{aligned}$$